DYNAMICS OF A POINT VORTEX AS LIMITS OF A SHRINKING SOLID IN AN IRROTATIONAL FLUID

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Abstract. We consider the motion of a rigid body immersed in a two-dimensional perfect fluid. The fluid is assumed to be irrotational and confined in a bounded domain. We prove that when the body shrinks to a pointwise massless particle with fixed circulation, its dynamics in the limit is given by the point vortex equation.

As a byproduct of our analysis we also prove that when the body shrinks with a fixed mass the limit equation is a second-order differential equation involving a Kutta-Joukowski-type lift force, which extends the result of [7] to the case where the domain occupied by the solid-fluid system is bounded.

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1. Introduction

The vortex point system is a classical playground which originates from fluid mechanics and goes back to Helmholtz, Kirchoff, Routh, and Lin. It appeared as a idealized model where the vorticity of an ideal incompressible two-dimensional fluid is concentrated in a finite number of points. Despite the fact that it does not constitute a solution of the Euler equations in the sense of distributions, it is now well-known that point vortices can be viewed as limits of concentrated smooth vortices which evolve according to the Euler equations. In the case of a single vortex moving in a bounded and simply-connected domain this was proved by Turkington in [22]. An extension to the case of several vortices was given by Marchioro and Pulvirenti, see [15]. Recently Gallay has proven in [6] that the vortex point system can also be obtained as vanishing viscosity limits of concentrated smooth vortices evolving according to the incompressible Navier-Stokes equations.

The main goal of this paper is to prove that the vortex point system can also be viewed as the limit of the dynamics of a solid, shrinking into a pointwise massless particle with fixed circulation, in free motion in an irrotational fluid.

Actually our analysis also allows to cover another asymptotic regime corresponding to the shrinking of a solid with fixed mass and circulation. In this case we obtain at the limit a second-order differential equation involving a Kutta-Joukowski-type lift force, which extends the result of [7] to the case where the solid-fluid system is bounded. Indeed this second case is way easier to tackle and we will therefore present it first in the sequel as a warm up before the difficulties appearing in the massless case.

We will consider the same setting than Turkington in [22], that is we assume that the fluid is ideal, confined in a two-dimensional bounded domain and we consider the motion of a single solid immersed in it. Moreover the flow is supposed to be irrotational.

We are interested in determining the limit of the dynamics of the solid when its size goes to 0, distinguishing two cases:

- Case (i): when the mass of the solid is fixed (and then the solid tends to a mass pointwise particle), and
- Case (ii): when the mass tends to 0 along with the size (and then the solid tends to a massless pointwise particle). This encompasses the case of fixed density.

1.1. **Dynamics of a solid with fixed size and mass.** To begin with, let us recall the dynamics of a solid with fixed size and mass. We denote by Ω the bounded open regular connected and simply connected domain of $\mathbb{R}^2$ occupied by the system fluid-solid.

We assume without loss of generality that

$$0 \in \Omega,$$

and that the logarithmic capacity[1] $\mathrm{Cap}_{\partial\Omega}$ of $\partial\Omega$ satisfies

$$\mathrm{Cap}_{\partial\Omega} < 1,$$

using translation and dilatation of the coordinates system if necessary. At the initial time, the domain of the solid is a non-empty closed regular connected and simply connected set $\mathcal{S}_0 \subset \Omega$ and

$$\mathcal{F}_0 := \Omega \setminus \bar{\mathcal{S}}_0,$$

[1] also called external conformal radius or transfinite diameter in other contexts [21]

is the domain of the fluid. Observe that the monotony property of the logarithmic capacity entails that $\mathrm{Cap}_{\partial\mathcal{S}_0} < 1$.

There is no loss of generality in assuming that the center of mass of the solid coincides at the initial time with the origin.

The rigid motion of the solid is described at every moment by a rotation matrix

$$R(\vartheta(t)) := \begin{bmatrix} \cos\vartheta(t) & -\sin\vartheta(t) \\ \sin\vartheta(t) & \cos\vartheta(t) \end{bmatrix},$$

describing the rotation of the solid with respect to its original position and a vector $h(t) \in \mathbb{R}^2$ describing the position of the center of mass. The domain of the solid at every time $t > 0$ is therefore

$$\mathcal{S}(t) := R(\vartheta(t))\mathcal{S}_0 + h(t),$$

while the domain of the fluid is

$$\mathcal{F}(t) := \Omega \setminus \bar{\mathcal{S}}(t).$$

The fluid-solid system is governed by the following set of coupled equations:

Fluid equations:

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u + \nabla\pi = 0 \quad \text{in } \mathcal{F}(t), \tag{1.1a}$$

$$\operatorname{div} u = 0 \quad \text{in } \mathcal{F}(t), \tag{1.1b}$$

Solid equations:

$$\vartheta' = \omega, \quad h' = \ell, \tag{1.1c}$$

$$m\ell' = \int_{\partial\mathcal{S}(t)} \pi n \,\mathrm{d}s, \tag{1.1d}$$

$$\mathcal{J}\omega' = \int_{\partial\mathcal{S}(t)} (x - h(t))^\perp \cdot \pi n \,\mathrm{d}s, \tag{1.1e}$$

Boundary conditions:

$$u\cdot n = \big(\omega(\cdot - h)^\perp + \ell\big)\cdot n \quad \text{on } \partial\mathcal{S}(t), \tag{1.1f}$$

$$u\cdot n = 0 \quad \text{on } \partial\Omega, \tag{1.1g}$$

Initial data:

$$u_{t=0} = u_0 \quad \text{in } \mathcal{F}_0, \tag{1.1h}$$

$$\vartheta(0) = 0,\ h(0) = 0,\ \ell(0) = \ell_0,\ \omega(0) = \omega_0. \tag{1.1i}$$

Above $u = (u_1, u_2)$ and π denote the velocity and pressure fields in the fluid, $m > 0$ and $\mathcal{J} > 0$ denote respectively the mass and the moment of inertia of the body while the fluid is supposed to be homogeneous of density 1, to simplify the notations. When $x = (x_1, x_2)$ the notation $x^\perp$ stands for $x^\perp = (-x_2, x_1)$, n denotes the unit normal vector pointing outside the fluid, $\ell(t) = h'(t)$ is the velocity of the center of mass $h(t) \in \mathbb{R}^2$ of the body and $\omega(t) \in \mathbb{R}$ denotes the angular velocity of the rigid body at time t. Let us also emphasize that we will use $\mathrm{d}s$ as surface element without any distinction on $\partial\Omega$, $\partial\mathcal{S}(t)$ and on $\partial\mathcal{S}_0$.

Let us recall that if the flow is irrotational at the initial time, that is if $\operatorname{curl} u_0 = 0$ in $\mathcal{F}_0$, it will remain irrotational for every time, that is

$$\operatorname{curl} u(t,\cdot) = 0 \text{ in } \mathcal{F}(t), \tag{1.2}$$

according to Helmholtz's third theorem. On the other hand the circulation around the body is constant in time:

$$\int_{\partial\mathcal{S}(t)} u(t)\cdot\tau \mathrm{d}s = \gamma, \tag{1.3}$$

with

$$\gamma = \int_{\partial\mathcal{S}_0} u_0\cdot\tau \mathrm{d}s,$$

according to Kelvin's theorem. Here τ denotes the unit counterclockwise tangential vector so that $n = \tau^\perp$. Let us mention here that we will also use the notation τ on $\partial\Omega$ such that $n := \tau^\perp$ so that it is clockwise.

In the irrotational case, the system (1.1) can be recast as an ODE whose unknowns are the degrees of freedom of the solid, namely ϑ and h. In particular the motion of the fluid is completely determined by the solid position and velocity. In order to state this, let us introduce the variables

$$h := (h_1, h_2), \quad q := (\vartheta, h_1, h_2) \in \mathbb{R}^3,$$

and their time derivatives

$$\ell := (\ell_1, \ell_2), \quad p := (\omega, \ell_1, \ell_2) \in \mathbb{R}^3.$$

Since the domains $\mathcal{S}(t)$ and $\mathcal{F}(t)$ depend on q only, we shall rather denote them $\mathcal{S}(q)$ and $\mathcal{F}(q)$ in the rest of the paper. Since throughout this paper we will not consider any collision, we introduce:

$$\mathcal{Q} := \{q \in \mathbb{R}^3 \, : \, d(\mathcal{S}(q), \partial\Omega) > 0\}, \tag{1.4}$$

where $d(A, B)$ denotes for two sets A and B in the plane

$$d(A, B) := \min\{|x - y|_{\mathbb{R}^2}, \ x \in A, \ y \in B\}.$$

Above and all along the paper we will use the notation $|\cdot|_{\mathbb{R}^d}$ for the Euclidean norm in $\mathbb{R}^d$. Since $\mathcal{S}_0$ is a closed subset in the open set Ω the initial position $q(0) = 0$ of the solid belongs to $\mathcal{Q}$.

Now we will need to introduce various objects depending on the geometry and on m, $\mathcal{J}$, γ, in order to make the ODE explicit.

Kirchhoff potentials. Consider the functions ξ_j, for $j = 1, 2, 3$, defined for $(q, x) \in \cup_{q\in\mathcal{Q}} \Big(\{q\} \times \mathcal{F}(q)\Big)$, by the formula

$$\xi_1(q, x) := (x - h)^\perp \text{ and } \xi_j(q, x) := e_{j-1}, \text{ for } j = 2, 3. \tag{1.5}$$

Above e_1 and e_2 are the unit vectors of the canonical basis. For any $j = 1, 2, 3$, for any q in $\mathcal{Q}$, we denote by $K_j(q, \cdot)$ the normal trace of ξ_j on $\partial\Omega \cup \partial\mathcal{S}(q)$, that is:

$$K_j(q, \cdot) := n \cdot \xi_j(q, \cdot) \text{ on } \partial\Omega \cup \partial\mathcal{S}(q). \tag{1.6}$$

We introduce the Kirchhoff's potentials $\varphi_j(q, \cdot)$, for $j = 1, 2, 3$, which are the unique (up to an additive constant) solutions in $\mathcal{F}(q)$ of the following Neumann problem:

$$\Delta\varphi_j = 0 \qquad \text{in } \mathcal{F}(q), \tag{1.7a}$$

$$\frac{\partial\varphi_j}{\partial n}(q, \cdot) = K_j(q, \cdot) \qquad \text{on } \partial\mathcal{S}(q), \tag{1.7b}$$

$$\frac{\partial\varphi_j}{\partial n}(q, \cdot) = 0 \qquad \text{on } \partial\Omega. \tag{1.7c}$$

We also denote

$$\boldsymbol{K}(q,\cdot) := (K_1(q,\cdot), K_2(q,\cdot), K_3(q,\cdot))^t \text{ and } \boldsymbol{\varphi}(q,\cdot) := (\varphi_1(q,\cdot), \varphi_2(q,\cdot), \varphi_3(q,\cdot))^t, \tag{1.8}$$

where the exponent t denotes the transpose of the vector.

Inertia matrices and impulses. We can now define the mass matrices

$$M_g := \begin{pmatrix} \mathcal{J} & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}, \tag{1.9a}$$

$$M_a(q) := \int_{\partial\mathcal{S}(q)} \boldsymbol{\varphi}(q,\cdot) \otimes \frac{\partial \boldsymbol{\varphi}}{\partial n}(q,\cdot)\mathrm{d}s = \Big(\int_{\mathcal{F}(q)} \nabla\varphi_i \cdot \nabla\varphi_j \mathrm{d}x \Big)_{1\leqslant i,j\leqslant 3}, \tag{1.9b}$$

$$M(q) := M_g + M_a(q). \tag{1.9c}$$

The matrix $M(q)$ corresponds to the sum of the genuine inertia M_g of the body and the so-called added inertia $M_a(q)$, which, loosely speaking, measures how much the surrounding fluid resists the acceleration as the body moves through it (since it needs to be accelerated as well). Both M_g and $M_a(q)$ are symmetric and positive-semidefinite, and M_g is definite. The index "g" above stands for "genuine", and the index "a" for "added".

We also introduce the impulses:

$$\begin{pmatrix} \Pi_g \\ P_g \end{pmatrix} := M_g p, \quad \begin{pmatrix} \Pi_a \\ P_a \end{pmatrix} := M_a(q) p, \quad \begin{pmatrix} \Pi \\ P \end{pmatrix} = \begin{pmatrix} \Pi_g + \Pi_a \\ P_g + P_a \end{pmatrix}. \tag{1.10}$$

Christoffel symbols. We can then define the bilinear symmetric mapping $\Gamma(q)$ associated with $M(q)$ by the formula

$$\langle \Gamma(q), p, p\rangle := \Bigg(\sum_{1\leqslant i,j\leqslant 3} \Gamma^k_{i,j}(q) p_i p_j \Bigg)_{1\leqslant k\leqslant 3} \in \mathbb{R}^3, \tag{1.11a}$$

where, for every $i,j,k \in \{1,2,3\}$, we set

$$\Gamma^k_{i,j}(q) := \frac{1}{2}\Big((M_a)^i_{k,j} + (M_a)^j_{k,i} - (M_a)^k_{i,j} \Big)(q), \tag{1.11b}$$

where $(M_a)^k_{i,j}$ denotes the partial derivative with respect to q_k of the entry of indexes (i,j) of the matrix M_a, that is

$$(M_a)^k_{i,j} := \frac{\partial (M_a)_{i,j}}{\partial q_k}. \tag{1.11c}$$

With a slight imprecision, we call the coefficients $\Gamma^k_{i,j}$ the *Christoffel symbols* associated with the mass matrix. Usually, one should multiply by $M(q)^{-1}$ the right hand side (1.11b) considered as a column vector indexed by k to get the standard Christoffel symbols.

We underline that since the genuine inertia M_g of the body is independent of the position q of the solid, only the added inertia is involved in the Christoffel symbols.

Stream function for the circulation term. Now we can introduce a function ψ as follows. For every $q \in \mathcal{Q}$, there exists a unique $C(q) \in \mathbb{R}$ such that the unique solution $\psi(q,\cdot)$ of the Dirichlet problem:

$$\Delta\psi(q,\cdot) = 0 \quad \text{in } \mathcal{F}(q) \tag{1.12a}$$

$$\psi(q,\cdot) = C(q) \quad \text{on } \partial\mathcal{S}(q) \tag{1.12b}$$

$$\psi(q,\cdot) = 0 \quad \text{on } \partial\Omega, \tag{1.12c}$$

satisfies

$$\int_{\partial \mathcal{S}(q)} \frac{\partial \psi}{\partial n}(q,\cdot)\mathrm{d}s = -1. \tag{1.12d}$$

This can be seen easily by defining the corresponding harmonic function $\tilde{\psi}(q,\cdot)$ with $\tilde{\psi}(q,\cdot) = 1$ on $\partial\mathcal{S}(q)$ and $\tilde{\psi}(q,\cdot) = 0$ on $\partial\Omega$ and renormalizing it (using the strong maximum principle gives $\frac{\partial\tilde{\psi}}{\partial n}(q,\cdot) < 0$ on $\partial\mathcal{S}(q)$ so that $\int_{\partial\mathcal{S}(q)} \frac{\partial\tilde{\psi}}{\partial n}(q,\cdot)\mathrm{d}s < 0$).

The function $C(q)$ is actually the opposite of the inverse of the condenser capacity of $\mathcal{S}(q)$ in Ω, that is, of $\int_{\mathcal{F}(q)} |\nabla\tilde{\psi}(q,\cdot)|^2\,\mathrm{d}x$. Observe that

$$\forall q \in \mathcal{Q}, \quad C(q) = -\int_{\mathcal{F}(q)} |\nabla\psi(q,\cdot)|^2\,\mathrm{d}x < 0, \tag{1.13}$$

$$C \in C^\infty(\mathcal{Q};(-\infty,0)) \text{ and depends on } \mathcal{S}_0 \text{ and } \Omega. \tag{1.14}$$

Regarding (1.14) and similar properties below of regularity with respect to shape, we refer to [11].

Force term. Eventually, we also define:

$$B(q) := \int_{\partial\mathcal{S}(q)} \left(\frac{\partial\psi}{\partial n}\left(\frac{\partial\boldsymbol{\varphi}}{\partial n} \times \frac{\partial\boldsymbol{\varphi}}{\partial\tau}\right)\right)(q,\cdot)\,\mathrm{d}s, \tag{1.15a}$$

$$E(q) := -\frac{1}{2}\int_{\partial\mathcal{S}(q)} \left(\left|\frac{\partial\psi}{\partial n}\right|^2 \frac{\partial\boldsymbol{\varphi}}{\partial n}\right)(q,\cdot)\,\mathrm{d}s, \tag{1.15b}$$

and the force term

$$F(q,p) := \gamma^2 E(q) + \gamma\, p \times B(q). \tag{1.15c}$$

We recall that γ denotes the circulation around the body.

Remark 1. *The notations E and B are chosen on purpose to highlight the analogy with the Lorentz force acting on a charged particle moving under the influence of a couple of electromagnetic fields E and B. This force vanishes if $\gamma = 0$.*

It can be checked that

$$M \in C^\infty(\mathcal{Q};S_3^{++}(\mathbb{R})) \text{ and depends on } \mathcal{S}_0, m, \mathcal{J} \text{ and } \Omega, \tag{1.16a}$$

$$F \in C^\infty(\mathcal{Q}\times\mathbb{R}^3;\mathbb{R}^3) \text{ and depends on } \mathcal{S}_0, \gamma \text{ and } \Omega, \text{ and vanishes when } \gamma = 0, \tag{1.16b}$$

$$\Gamma \in C^\infty(\mathcal{Q};\mathcal{BL}(\mathbb{R}^3\times\mathbb{R}^3;\mathbb{R}^3)) \text{ and depends on } \mathcal{S}_0 \text{ and } \Omega. \tag{1.16c}$$

Above $S_3^{++}(\mathbb{R})$ denotes the set of real symmetric positive-definite 3×3 matrices, $\mathcal{BL}(\mathbb{R}^3\times\mathbb{R}^3;\mathbb{R}^3)$ denotes the space of bilinear mappings from $\mathbb{R}^3\times\mathbb{R}^3$ to $\mathbb{R}^3$.

We stress that M does not depend on the circulation γ whereas F does not depend on m and $\mathcal{J}$ and Γ does not depend on m, γ and $\mathcal{J}$. In the following, when specifying these dependences is relevant, we will denote

$$M[\mathcal{S}_0, m, \mathcal{J}, \Omega],\ \Gamma[\mathcal{S}_0,\Omega] \text{ and } F[\mathcal{S}_0,\gamma,\Omega] \text{ instead of } M,\ \Gamma \text{ and } F. \tag{1.17}$$

Now our first result gives a reformulation of the system in terms of an ordinary differential equation.

Theorem 1. *Up to the first collision, System 1.1 is equivalent to the second order ODE*

$$q' = p, \tag{1.18a}$$

$$M(q)p' + \langle \Gamma(q), p, p \rangle = F(q,p), \tag{1.18b}$$

with Cauchy data

$$q(0) = 0 \in \mathcal{Q}, \ p(0) = (\omega_0, \ell_0) \in \mathbb{R} \times \mathbb{R}^2.$$

The proof of Theorem 1 is postponed to Section 9.

Remark 2. *Note that if $\gamma = 0$ and one faces the potential case, then the ODE* (1.18) *means that the particle is moving along the geodesics associated with the Riemann metric induced on $\mathcal{Q}$ by the matrix $M(q)$.*

According to classical ODE theory there exists a maximal time $T > 0$ and $q \in C^\infty([0,T);\mathcal{Q})$ a unique solution to (1.18) with Cauchy data $q(0) = (0,0)$, $p(0) = (\omega_0, \ell_0)$ and given γ.

Moreover, it follows from Corollary 1 below that T is the time of the first collision of the solid with the outer boundary of the fluid domain. If there is no collision, then $T = +\infty$.

Let us now turn our attention to the limit of the dynamics when the size of the solid goes to 0. As mentioned above, we will distinguish two cases:

- Case (i): when the mass of the solid is fixed (and then the solid tends to a massive pointwise particle), and
- Case (ii): when the mass tends to 0 along with the size (and then the solid tends to a massless pointwise particle).

1.2. **Case (i): Dynamics of a solid shrinking to a pointwise massive particle.** For every $\varepsilon \in (0,1]$, we denote

$$\mathcal{S}_0^\varepsilon := \varepsilon \mathcal{S}_0, \tag{1.19}$$

and for every $q = (\vartheta, h) \in \mathbb{R}^3$,

$$\mathcal{S}^\varepsilon(q) := R(\vartheta)\mathcal{S}_0^\varepsilon + h \text{ and } \mathcal{F}^\varepsilon(q) = \Omega \setminus \bar{\mathcal{S}}^\varepsilon(q). \tag{1.20}$$

The solid occupying the domain $\mathcal{S}^\varepsilon(q)$ is assumed to have a mass and a moment of inertia of the form

$$m^\varepsilon = m \text{ and } \mathcal{J}^\varepsilon = \varepsilon^2 \mathcal{J}^1, \tag{1.21}$$

where $m > 0$ and $\mathcal{J}^1 > 0$ are fixed.

With these settings, we denote by $(q^\varepsilon, p^\varepsilon)$ the solution to the ODE (1.18) associated with $M^\varepsilon := M[\mathcal{S}_0^\varepsilon, m^\varepsilon, \mathcal{J}^\varepsilon, \Omega]$, $\Gamma^\varepsilon := \Gamma[\mathcal{S}_0^\varepsilon, \Omega]$ and $F^\varepsilon := F[\mathcal{S}_0^\varepsilon, \gamma, \Omega]$ in place of M, Γ and F, respectively, defined on the maximal time interval $[0, T^\varepsilon)$. We decompose q^ε into

$$q^\varepsilon = (\vartheta^\varepsilon, h^\varepsilon) \in \mathbb{R} \times \mathbb{R}^2.$$

Notice that γ and the Cauchy data (p_0, q_0) are not depending on ε. The latter are decomposed into

$$p_0 = (\omega_0, \ell_0) \text{ and } q_0 = (0,0).$$

Our first result is the convergence, in this setting, of h^ε to the solution of a massive point vortex equation. Let us introduce this limit equation. Let $(h_{(i)}, T_{(i)})$ be the maximal solution of the ODE:

$$m(h_{(i)})'' = \gamma\big((h_{(i)})' - \gamma u_\Omega(h_{(i)})\big)^\perp \quad \text{for } t \in [0, T_{(i)}), \quad \text{with } h_{(i)}(0) = 0 \text{ and } (h_{(i)})'(0) = \ell_0, \tag{1.22}$$

where u_Ω is the Kirchhoff-Routh velocity defined as follows. We introduce $\psi^0_{\cancel{\partial\mathcal{S}}}(h,\cdot)$ as the solution of the following Dirichlet problem:

$$\Delta\psi^0_{\cancel{\partial\mathcal{S}}}(h,\cdot) = 0 \text{ in } \Omega, \quad \psi^0_{\cancel{\partial\mathcal{S}}}(h,\cdot) = G(\cdot - h) \text{ on } \partial\Omega, \tag{1.23}$$

where

$$G(r) := -\frac{1}{2\pi}\ln|r|. \tag{1.24}$$

The Kirchhoff-Routh stream function ψ_Ω is defined as

$$\psi_\Omega(x) := \frac{1}{2}\psi^0_{\cancel{\partial\mathcal{S}}}(x,x), \tag{1.25}$$

and the Kirchhoff-Routh stream velocity u_Ω is defined by

$$u_\Omega := \nabla^\perp\psi_\Omega, \tag{1.26}$$

where $\nabla^\perp := (-\partial_2, \partial_1)$.

The existence of $(h_{(i)}, T_{(i)})$ follows from classical ODE theory. Moreover it follows from the energy conservation stated below in (2.24) and from the continuity of the Kirchhoff-Routh stream function ψ_Ω in Ω that $T_{(i)}$ is the time of the first collision of $h_{(i)}$ with the outer boundary $\partial\Omega$ of the fluid domain. If there is no collision, then $T_{(i)} = +\infty$.

The precise statement of our first convergence result is as follows.

Theorem 2. *Let*

- $\mathcal{S}_0 \subset \Omega$*;*
- $p_0 \in \mathbb{R}^3$ *and* $(\gamma, m, \mathcal{J}) \in \mathbb{R}\times(0,+\infty)\times(0,+\infty)$*;*
- $(h_{(i)}, T_{(i)})$ *be the maximal solution of* (1.22)*;*
- *for every* $\varepsilon \in (0,1]$ *small enough to ensure that* $\mathcal{S}_0^\varepsilon \subset \Omega$, $((q^\varepsilon, p^\varepsilon), T^\varepsilon)$ *be the maximal solution of* (1.18) *with*

$$M^\varepsilon = M[\mathcal{S}_0^\varepsilon, m^\varepsilon, \mathcal{J}^\varepsilon, \Omega],\ \Gamma^\varepsilon = \Gamma[\mathcal{S}_0^\varepsilon, \Omega] \text{ and } F^\varepsilon = F[\mathcal{S}_0^\varepsilon, \gamma, \Omega]$$
$$\text{in place of } M,\ \Gamma \text{ and } F, \text{ respectively,}$$

where $\mathcal{S}_0^\varepsilon$ *is given by* (1.19) *and* $m^\varepsilon, \mathcal{J}^\varepsilon$ *are given by* (1.21)*, and with the initial data*

$$(q^\varepsilon, p^\varepsilon)(0) = (0, p_0).$$

Then, as $\varepsilon \to 0$*,*

- $\liminf T^\varepsilon \geqslant T_{(i)}$*,*
- $h^\varepsilon \longrightarrow h_{(i)}$ *in* $W^{2,\infty}([0,T];\mathbb{R}^2)$ *weak-*$\star$ *for all* $T \in (0, T_{(i)})$*,*
- $\varepsilon\vartheta^\varepsilon \longrightarrow 0$ *in* $W^{2,\infty}([0,T];\mathbb{R})$ *weak-*$\star$ *for all* $T \in (0, T_{(i)})$*.*

1.3. **Case (ii): Dynamics of a solid shrinking to a pointwise massless particle.** In this section the solid is still assumed to occupy initially the domain $\mathcal{S}_0^\varepsilon$ given by (1.19) but we assume now that it has a mass and a moment of inertia given by

$$m^\varepsilon = \alpha_\varepsilon m^1 \text{ and } \mathcal{J}^\varepsilon = \alpha_\varepsilon\varepsilon^2\mathcal{J}^1, \tag{1.27}$$

where $\alpha_\varepsilon \to 0$ when $\varepsilon \to 0$ and $\alpha_1 = 1$, and where $m^1 > 0$ and $\mathcal{J}^1 > 0$ are fixed. In order to simplify the notations we will assume that α_ε is of the form

$$\alpha_\varepsilon = \varepsilon^\alpha,$$

with $\alpha > 0$. The particular case where $\alpha = 2$ corresponds to the case of a fixed solid density. Case (i) corresponded to the case where $\alpha = 0$.

In this setting, we denote by $(p^\varepsilon, q^\varepsilon)$ the solution to the ODE (1.18) defined on the time interval $[0, T^\varepsilon)$. Let us stress that the circulation γ and the Cauchy data are still assumed not to depend on ε. Moreover we will assume that

$$\gamma \neq 0.$$

Our second result is the convergence of h^ε to the solution of the point vortex equation:

$$(h_{(ii)})' = \gamma u_\Omega(h_{(ii)}) \text{ for } t > 0, \text{ with } h_{(ii)}(0) = 0. \tag{1.28}$$

It is well-known that the solution $h_{(ii)}$ is global in time, and in particular that there is no collision of the vortex point with the external boundary $\partial\Omega$. This follows from (2.25) below and the fact that $\psi_\Omega(h) \to +\infty$ when h comes close to $\partial\Omega$, see for instance [22, Eq. (1.27)].

More precisely, our result is the following.

Theorem 3. *Let*

- $\mathcal{S}_0 \subset \Omega$;
- $\gamma \neq 0$;
- $p_0 \in \mathbb{R}^3$ *and* $(\gamma, m, \mathcal{J}) \in \mathbb{R} \times (0, +\infty) \times (0, +\infty)$;
- $h_{(ii)}$ *be the global solution of* (1.28);
- *for every* $\varepsilon \in (0, 1]$ *small enough to ensure that* $\mathcal{S}_0^\varepsilon \subset \Omega$, $((q^\varepsilon, p^\varepsilon), T^\varepsilon)$ *be the maximal solution of* (1.18) *with*

$$M^\varepsilon = M[\mathcal{S}_0^\varepsilon, m^\varepsilon, \mathcal{J}^\varepsilon, \Omega],\ \Gamma^\varepsilon = \Gamma[\mathcal{S}_0^\varepsilon, \Omega] \text{ and } F^\varepsilon = F[\mathcal{S}_0^\varepsilon, \gamma, \Omega]$$
$$\text{in place of } M,\ \Gamma \text{ and } F, \text{ respectively,}$$

where $\mathcal{S}_0^\varepsilon$ *is given by* (1.19) *and* $m^\varepsilon, \mathcal{J}^\varepsilon$ *are given by* (1.27), *and with the initial data*

$$(q^\varepsilon, p^\varepsilon)(0) = (0, p_0).$$

Then, as $\varepsilon \to 0^+$,

- $T^\varepsilon \longrightarrow +\infty$,
- $h^\varepsilon \rightharpoonup h_{(ii)}$ *in* $W^{1,\infty}([0, T]; \mathbb{R}^2)$ *weak-*$\star$ *for all* $T > 0$.

1.4. **A few comments.** Let us emphasize that the limit systems obtained in Case (i) and in Case (ii) do not depend on the body shape nor on the value of $\alpha > 0$. Still the proof is more simple in the case where the body is a disk. Indeed if $\mathcal{S}_0$ is a disk, in both Cases (i) and Case (ii), it follows directly from (1.1e) that the rotation ϑ^ε satisfies, for any $\varepsilon \in (0, 1)$, $\vartheta^\varepsilon(t) = t\omega_0$ as long as the solution exists.

One may wonder if the weak-$\star$ convergence obtained in Theorem 2 and Theorem 3 can be improved. In general it seems that some strong oscillations in time show up when $\varepsilon \to 0$ which prevent a strong convergence. We plan to study this phenomenon by a multi-scale approach of the solution of the ODE (1.18) in a forthcoming work. Once again the case where the body is a disk is likely to simplify the discussion.

In Case (i), one may also raise the question whether it is possible that $\liminf T^\varepsilon > T_{(i)}$. This problem should be connected to the behaviour of the potentials and stream functions as the body approaches the boundary; see for instance [3], [4] and [17] and references therein for this question.

The analysis performed in this paper can be easily adapted in order to cover the case where the circulation γ depends on ε under the form $\gamma^\varepsilon = \varepsilon^\beta\, \gamma^1$ with $\beta > 0$ in Case (i) and $\beta \in (0, 1)$ in Case (ii). Then one obtains respectively at the limit the trivial equations $(h_{(i)})'' = 0$ and $(h_{(ii)})' = 0$.

Our analysis should hold as well in the case of several bodies moving in the full plane or in a multiply-connected domain, as long as there is no collision. This will be tackled in a forthcoming work.

Another natural question is whether or not one may extend the results of Theorem 2 and Theorem 3 to rotational flows. This issue is tackled in a more restricted geometric set-up in the work in preparation [8].

2. Preliminary material

In this section, we introduce some material that will be important in the sequel.

We begin by introducing some notations.
Let

- for $\delta > 0$,

$$\mathcal{Q}_\delta := \{q \in \mathbb{R}^3 \, : \, d(\mathcal{S}(q), \partial\Omega) > \delta\}, \tag{2.1}$$

- for $\varepsilon > 0$,

$$\mathcal{Q}^\varepsilon := \{q \in \mathbb{R}^3 \, : \, d(\mathcal{S}^\varepsilon(q), \partial\Omega) > 0\}, \quad \mathfrak{Q} := \cup_{\varepsilon\in[0,1)} \Big(\{\varepsilon\} \times \mathcal{Q}^\varepsilon\Big), \tag{2.2}$$

- for $\delta > 0$,

$$\mathcal{Q}^\varepsilon_\delta := \{q \in \mathbb{R}^3 \, : \, d(\mathcal{S}^\varepsilon(q), \partial\Omega) > \delta\}, \quad \mathfrak{Q}_\delta := \cup_{\varepsilon\in[0,1)} \Big(\{\varepsilon\} \times \mathcal{Q}^\varepsilon_\delta\Big), \tag{2.3}$$

- for $\delta > 0$ and $\varepsilon_0 \in (0, 1)$,

$$\mathfrak{Q}_{\delta,\varepsilon_0} := \{(\varepsilon, q) \in \mathfrak{Q}_\delta / \ \varepsilon \in (0, \varepsilon_0)\}. \tag{2.4}$$

We will also make use, for $\delta > 0$, of

$$\Omega_\delta := \{x \in \Omega / \quad d(x, \partial\Omega) > \delta\}. \tag{2.5}$$

Observe that despite the fact that center of mass h^ε does not necessarily belong to $\mathcal{S}^\varepsilon(q)$, we have the following.

Lemma 1. *Let $\delta > 0$. There exists $\delta_0 \in (0, \delta)$ and $\varepsilon_0 \in (0, 1]$ such that for any $(\varepsilon, q) \in \mathfrak{Q}_{\delta,\varepsilon_0}$, with $q = (\vartheta, h)$, necessarily $h \in \Omega_{\delta_0}$.*

Proof of Lemma 1. Let us introduce $R_0 := \max\{|x|, \ x \in \partial\mathcal{S}_0\}$. Set $\delta_0 := \frac{\delta}{2}$ and $\varepsilon_0 := \min(1, \frac{\delta}{2R_0})$. Let $(\varepsilon, q) \in \mathfrak{Q}_{\delta,\varepsilon_0}$, with $q = (\vartheta, h)$. We pick $\underline{h} \in \mathcal{S}^\varepsilon(q)$; in particular $d(\underline{h}, \partial\Omega) > \delta$. Since $\mathcal{S}^\varepsilon(q)$ is a subset of the closed disk $\overline{B}(h, \varepsilon R_0)$ of center h and of radius εR_0, we have $d(h, \underline{h}) \leqslant \varepsilon R_0 \leqslant \frac{\delta}{2}$. Then observing that $d(h, \partial\Omega) \geqslant d(\underline{h}, \partial\Omega) - d(\underline{h}, h) \geqslant \delta_0$, we deduce the claim. $\square$

2.1. Dynamics of the solid without external boundary.

Theorem 1 extends to the case of a bounded domain Ω a result which is well-known in the case where the domain Ω occupied by the fluid-solid system is the plane whole, i.e. $\Omega = \mathbb{R}^2$, with the fluid at rest at infinity. And it turns out that the objects associated with this case are of central importance in our analysis.

The equations (1.18) when $\Omega = \mathbb{R}^2$ are as follows:

$$q' = p, \quad M_{\cancel{\partial\Omega},\vartheta} \, p' + \langle \Gamma_{\cancel{\partial\Omega},\vartheta}, p, p\rangle = F_{\cancel{\partial\Omega},\vartheta}(p), \tag{2.6}$$

where

$$M_{\cancel{\partial\Omega},\vartheta} := M[\mathcal{S}_0, m^1, \mathcal{J}^1, \mathbb{R}^2](q), \ \ \Gamma_{\cancel{\partial\Omega},\vartheta} := \Gamma[\mathcal{S}_0, \mathbb{R}^2](q) \text{ and } F_{\cancel{\partial\Omega},\vartheta}(p) := F[\mathcal{S}_0, \gamma, \mathbb{R}^2](q, p). \tag{2.7}$$

Observe in particular that the dependence on q of M, Γ and F reduces to a dependence on the rotation $R(\vartheta)$ only; from now on, we will mention this dependence on ϑ through an index, so q does no longer appear as an argument.

Of course, as previously, M, Γ and F also depend on $\mathcal{S}_0$, m^1, $\mathcal{J}^1$ and γ, but here this dependence can be made rather explicit. We describe below the form of these functions.

Kirchhoff potential and inertia matrices. Let us first denote by $\varphi_{\cancel{\partial\Omega},j}$, for $j = 1, 2, 3$, the Kirchhoff's potentials in $\mathbb{R}^2 \setminus \mathcal{S}_0$ which are the functions that satisfy the following Neumann problem:

$$\Delta\varphi_{\cancel{\partial\Omega},j} = 0 \qquad \text{in } \mathbb{R}^2 \setminus \mathcal{S}_0, \tag{2.8a}$$

$$\frac{\partial\varphi_{\cancel{\partial\Omega},j}}{\partial n} = x^\perp \cdot n \quad \text{for } j = 1 \qquad \text{on } \partial\mathcal{S}_0, \tag{2.8b}$$

$$\frac{\partial\varphi_{\cancel{\partial\Omega},j}}{\partial n} = e_{j-1} \cdot n, \quad \text{for } j = 2, 3 \qquad \text{on } \partial\mathcal{S}_0,$$

$$\nabla\varphi_{\cancel{\partial\Omega},j}(x) \to 0 \qquad \text{at infinity.} \tag{2.8c}$$

We also denote

$$\boldsymbol{\varphi}_{\cancel{\partial\Omega}} := (\varphi_{\cancel{\partial\Omega},1},\, \varphi_{\cancel{\partial\Omega},2},\, \varphi_{\cancel{\partial\Omega},3})^t. \tag{2.9}$$

We can now define the added mass matrix

$$M_{a,\cancel{\partial\Omega}} := \int_{\partial\mathcal{S}_0} \boldsymbol{\varphi}_{\cancel{\partial\Omega}} \otimes \frac{\partial\boldsymbol{\varphi}_{\cancel{\partial\Omega}}}{\partial n} \mathrm{d}s = \Big(\int_{\mathbb{R}^2\setminus\mathcal{S}_0} \nabla\varphi_{\cancel{\partial\Omega},i} \cdot \nabla\varphi_{\cancel{\partial\Omega},j} \mathrm{d}x \Big)_{1\leqslant i,j\leqslant 3}. \tag{2.10}$$

Let us notice that the matrix $M_{a,\cancel{\partial\Omega}}$ is symmetric positive-semidefinite and depends only on $\mathcal{S}_0$. Actually it is positive definite if and only if $\mathcal{S}_0$ is not a disk. Moreover, when $\mathcal{S}_0$ is a disk, the matrix $M_{a,\cancel{\partial\Omega}}$ is diagonal, of the form $M_{a,\cancel{\partial\Omega}} := \operatorname{diag}(0, m_{a,\cancel{\partial\Omega}}, m_{a,\cancel{\partial\Omega}})$ with $m_{a,\cancel{\partial\Omega}} > 0$.

Then we can introduce the mass matrix $M1_{\cancel{\partial\Omega},\vartheta}(q)$ taking the rotation into account by

$$M^1_{\cancel{\partial\Omega},\vartheta} := M^1_g + M_{a,\cancel{\partial\Omega},\vartheta}, \quad M^1_g := \begin{pmatrix} \mathcal{J}^1 & 0 & 0 \\ 0 & m^1 & 0 \\ 0 & 0 & m^1 \end{pmatrix} \text{ and } M_{a,\cancel{\partial\Omega},\vartheta} := \mathcal{R}(\vartheta) M_{a,\cancel{\partial\Omega}} \mathcal{R}(\vartheta)^t. \tag{2.11}$$

Above we used the notation

$$\mathcal{R}(\vartheta) := \begin{pmatrix} 1 & 0 \\ 0 & R(\vartheta) \end{pmatrix} \in \mathrm{SO}(3). \tag{2.12}$$

Christoffel symbols. Without outer boundary, the Christoffel symbols take the simple following form:

$$\langle \Gamma_{\cancel{\partial\Omega},\vartheta}, p, p \rangle := - \begin{pmatrix} 0 \\ P_{a,\cancel{\partial\Omega},\vartheta} \end{pmatrix} \times p - \omega M_{a,\cancel{\partial\Omega},\vartheta} \begin{pmatrix} 0 \\ \ell^\perp \end{pmatrix}, \tag{2.13}$$

where $P_{a,\cancel{\partial\Omega},\vartheta}$ denotes the last two coordinates of $M_{a,\cancel{\partial\Omega},\vartheta}\, p$.

Force term. The force term $F_{\cancel{\partial\Omega},\vartheta}(p)$ in that case is given by

$$F_{\cancel{\partial\Omega},\vartheta}(p) := \gamma \begin{pmatrix} \zeta_\vartheta \cdot \ell \\ \ell^\perp - \omega\zeta_\vartheta \end{pmatrix}, \tag{2.14}$$

where

$$\zeta_\vartheta := R(\vartheta)\zeta, \tag{2.15}$$

and the geometric constant $\zeta \in \mathbb{R}^2$, depending only on $\mathcal{S}_0$ is defined as follows. In the same spirit as (1.12), we first introduce the function $\psi^{-1}_{\cancel{\partial\Omega}}$ as the solution of

$$-\Delta\psi^{-1}_{\cancel{\partial\Omega}} = 0 \quad \text{in } \mathbb{R}^2 \setminus \mathcal{S}_0, \tag{2.16a}$$

$$\psi^{-1}_{\cancel{\partial\Omega}} = C_{\cancel{\partial\Omega}} \quad \text{on } \partial\mathcal{S}_0, \tag{2.16b}$$

$$\psi^{-1}_{\cancel{\partial\Omega}} = O(\ln|x|) \quad \text{at infinity,} \tag{2.16c}$$

where the constant

$$C_{\cancel{\partial\Omega}} = -\frac{1}{2\pi}\ln\left(\frac{1}{\mathrm{Cap}_{\partial\mathcal{S}_0}}\right) \tag{2.16d}$$

is such that:

$$\int_{\partial\mathcal{S}_0} \frac{\partial\psi^{-1}_{\cancel{\partial\Omega}}}{\partial n}\,\mathrm{d}s = -1. \tag{2.16e}$$

The existence and uniqueness of $\psi^{-1}_{\cancel{\partial\Omega}}$ will be recalled below in Corollary 6.

Then ζ is defined by

$$\zeta := -\int_{\partial\mathcal{S}_0} x\frac{\partial\psi^{-1}_{\cancel{\partial\Omega}}}{\partial n}\,\mathrm{d}s. \tag{2.17}$$

and it is usually referred to as the conformal center of gravity of $\mathcal{S}_0$.

An important feature of the force $F_{\cancel{\partial\Omega},\vartheta}(p)$ is that it is gyroscopic, in the sense of the following definition, see for instance [1, p. 428].

Definition 1. *We say that a vector field $F \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R}^3)$ is gyroscopic if for any (q,p) in $\mathbb{R}^3 \times \mathbb{R}^3$, $p \cdot F(q,p) = 0$.*

Indeed, for any (ϑ, p) in $\mathbb{R} \times \mathbb{R}^3$, the force $F_{\cancel{\partial\Omega},\vartheta}(p)$ can be written as

$$F_{\cancel{\partial\Omega},\vartheta}(p) = \gamma p \times B_{\cancel{\partial\Omega},\vartheta}, \tag{2.18}$$

with

$$B_{\cancel{\partial\Omega},\vartheta} = \begin{pmatrix} -1 \\ \zeta_\vartheta^\perp \end{pmatrix}. \tag{2.19}$$

We used the following formula for the vector product, which will be also useful later on in some computations:

$$\forall p_a := (\omega_a, \ell_a),\ p_b := (\omega_b, \ell_b) \ \text{ in } \ \mathbb{R} \times \mathbb{R}^2, \quad p_a \times p_b = (\ell_a^\perp \cdot \ell_b,\ \omega_a\, \ell_b^\perp - \omega_b\, \ell_a^\perp). \tag{2.20}$$

Observe that, compared to (1.15c), there is no electric-type field in (2.18).

The derivation of (2.6) seems to date back at least to Lamb [14, Article 134a.]. With respect to Theorem 1, the analysis is simplified from a geometrical point of view, since considering a frame attached to the body allows to reduce the problem to a fixed boundary one. In order to overcome the geometrical difficulty in the proof Theorem 1, we extend the analysis performed by the second author in [19] in the case of a vanishing circulation. We end up with a less explicit expression of the force, see (1.15), as compared to the force term $F_{\cancel{\partial\Omega},\vartheta}(p)$ described in (2.14).

Remark 3. *Let us mention that the derivation of* (2.6) *can also be obtained with a different strategy, relying on the use of complex analysis and more particularly on Blasius' lemma. This approach is due to Kutta, Joukowski and Chaplygin. An elegant exposition is given in* [18, Article 9.53]*. In this latter approach the geometrical vector ζ appears as a complex integral. The link is given by the following lemma, whose proof is postponed to an appendix.*

Lemma 2. *Denote* $\nabla^\perp \psi_{\partial\Omega}^{-1} := (H_1, H_2)$ *and* $\zeta := (\zeta_1, \zeta_2)$. *Then*

$$\zeta_1 + i\zeta_2 = \int_{\partial \mathcal{S}_0} z(H_1 - iH_2)\, \mathrm{d}z.$$

2.2. **The role of the energy.** In this subsection, we discuss several aspects concerning the energy of the systems considered here. The energy plays a central role in our analysis.

Conservation of energy. An important feature of the system (1.18) is that it is conservative. More precisely, we have the following result.

Proposition 1. *For any* $(q,p) \in C^\infty([0,T]; \mathcal{Q} \times \mathbb{R}^3)$ *satisfying* (1.18) *one has*

$$\frac{d}{dt}\mathcal{E}(q,p) = 0, \qquad \text{where } \mathcal{E}(q,p) := \frac{1}{2} M(q)p \cdot p + U(q), \tag{2.21}$$

where the potential energy $U(q)$ *is given by*

$$U(q) := -\frac{1}{2}\gamma^2 C(q),$$

with $C(q)$ *given by* (1.12). *Moreover*

$$\forall q \in \mathcal{Q}, \quad E(q) = \frac{1}{2} DC(q). \tag{2.22}$$

Above the notation $DC(q)$ stands for the derivative of $C(q)$ with respect to q. Let us emphasize that the energy function $\mathcal{E}$ belongs to $C^\infty(\mathcal{Q} \times \mathbb{R}^3; \mathbb{R})$ and is the sum of two positive signs, see (1.14) and (1.16a). In addition to its dependence on q and p, the energy $\mathcal{E}$ depends on $\mathcal{S}_0, m, \mathcal{J}, \gamma$ and Ω.

If we assume that the body stays at distance at least $\delta > 0$ from the boundary we may infer a bound of the body velocity depending only on the data and on δ. Indeed we have the following immediate corollary of Proposition 1, (1.14) and (1.16a).

Corollary 1. *Let*
- $\mathcal{S}_0 \subset \Omega$, $p_0 \in \mathbb{R}^3$ *and* $(\gamma, m, \mathcal{J}) \in \mathbb{R} \times (0, +\infty) \times (0, +\infty)$;
- $\delta > 0$;
- $(q,p) \in C^\infty([0,T]; \mathcal{Q}_\delta \times \mathbb{R}^3)$ *satisfying* (1.18) *with the Cauchy data* $(q,p)(0) = (0, p_0)$.

Then there exists $K > 0$ *depending only on* $\mathcal{S}_0, \Omega, p_0, \gamma, m, \mathcal{J}, \delta$ *such that* $|p|_{\mathbb{R}^3} \leqslant K$ *on* $[0,T]$.

Let us refer here to [12] for an example of collision of a disk moving in a potential flow (that is in the case where $\gamma = 0$) with the fixed boundary of the fluid domain.

The case of the whole plane. In the case where the domain Ω occupied by the fluid-solid system is the plane whole, i.e. $\Omega = \mathbb{R}^2$, with the fluid at rest at infinity, the potential $C(q)$ degenerates to the geometric constant $C_{\partial\Omega}$. This, combined with (2.22), explains why there is no electric-type field in (2.18).

Another consequence of the degeneracy of the potential $C(q)$ into a geometric constant in that case is that the equivalent of Proposition 1 gives the conservation of the kinetic energy

$$\mathcal{E}_{\partial\Omega,\vartheta}(p) := \frac{1}{2} M_{\partial\Omega,\vartheta}\, p \cdot p. \tag{2.23}$$

Observe in particular that the meaningless constant $\frac{1}{2}\gamma^2 C_{\partial\Omega}$ has been discarded.

Energy for the limit equations. Let us now turn to the energy conservations for the limit equations. It is classical and elementary to see that for any $h \in C^\infty([0,T]; \Omega)$,

- satisfying (1.22) one has

$$\frac{d}{dt}\mathcal{E}_{(i)}(h,\ell)=0, \quad \text{with } \mathcal{E}_{(i)}(h,\ell):=\frac{1}{2}m\ell\cdot\ell-\gamma^2\psi_\Omega(h), \text{ with } \ell=h', \tag{2.24}$$

- satisfying (1.28) one has

$$\frac{d}{dt}\mathcal{E}_{(ii)}(h)=0, \quad \text{with } \mathcal{E}_{(ii)}(h):=\gamma^2\psi_\Omega(h). \tag{2.25}$$

Theorem 2 and Theorem 3 therefore respectively assert the convergence of the trajectories of the system (1.18) associated with the energy $\mathcal{E}(q,p)$ given by (2.21) to those of the Hamiltonian system (1.22) associated with the energy (2.24) in Case (i) and to the Hamiltonian system (1.28) associated with the energy (2.25) in Case (ii).

For further Hamiltonian aspects related to Systems (1.1) and (1.28), we refer for instance to [15, 20, 23].

2.3. **Scaling with respect to ε.** In this subsection, we give a few remarks concerning the scaling of several objects with respect to ε.

Genuine inertia matrix and kinetic energy. Under the relation (1.27), the matrix of genuine inertia reads

$$M_g^\varepsilon:=\begin{pmatrix}\mathcal{J}^\varepsilon & 0 & 0\\ 0 & m^\varepsilon & 0\\ 0 & 0 & m^\varepsilon\end{pmatrix}=\varepsilon^\alpha\begin{pmatrix}\varepsilon^2\mathcal{J}^1 & 0 & 0\\ 0 & m^1 & 0\\ 0 & 0 & m^1\end{pmatrix}. \tag{2.26}$$

Let us introduce

$$I_\varepsilon:=\begin{pmatrix}\varepsilon & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{2.27}$$

Now we can extract ε from the inertia matrix as follows:

$$M_g^\varepsilon=\varepsilon^\alpha I_\varepsilon M_g^1 I_\varepsilon, \text{ with } M_g^1:=\begin{pmatrix}\mathcal{J}^1 & 0 & 0\\ 0 & m^1 & 0\\ 0 & 0 & m^1\end{pmatrix}. \tag{2.28}$$

Hence it is natural to associate with $p^\varepsilon:=(\omega^\varepsilon,\ell^\varepsilon)^t$ the vector

$$\hat{p}^\varepsilon:=I_\varepsilon p^\varepsilon=\begin{pmatrix}\hat{\omega}^\varepsilon\\ \ell^\varepsilon\end{pmatrix} \quad \text{with} \quad \hat{\omega}^\varepsilon:=\varepsilon\omega^\varepsilon. \tag{2.29}$$

In particular the solid kinetic energy of the solid can be recast as

$$\frac{1}{2}M_g^\varepsilon p^\varepsilon\cdot p^\varepsilon=\frac{1}{2}\varepsilon^\alpha M_g^1\,\hat{p}^\varepsilon\cdot\hat{p}^\varepsilon. \tag{2.30}$$

Hence the natural counterpart to ℓ^ε for what concerns the angular velocity is rather $\varepsilon\omega^\varepsilon$ than ω^ε.

This can also be seen on the boundary condition (1.1f): when x belongs to $\partial\mathcal{S}^\varepsilon(t)$, the term $\omega^\varepsilon(x-h^\varepsilon)^\perp$ is of order $\varepsilon\omega^\varepsilon$ and is added to ℓ^ε.

Added inertia matrix. The matrix M_a^ε corresponding to the added inertia in Ω associated with the solid $\mathcal{S}^\varepsilon$ of size ε depends in an intricate way on ε, see Proposition 4 below. However in the case of an outer domain such as described in Subsection 2.1, the added inertia matrix behaves in a simple way.

Let $M^{\varepsilon}_{a,\cancel{\partial\Omega}}$ be the matrix defined as in (2.10) for the solid $\mathcal{S}^{\varepsilon}_0$, and let us recall that $M_{a,\cancel{\partial\Omega}}$ denotes the one corresponding to $\mathcal{S}_0$, that is for $\varepsilon = 1$. Then one easily sees after suitable scaling arguments that

$$M^{\varepsilon}_{a,\cancel{\partial\Omega}} = \varepsilon^2 I_\varepsilon M_{a,\cancel{\partial\Omega}} I_\varepsilon. \tag{2.31}$$

Other terms in the case without outer boundary. The other terms in (2.6) have also a simple scaling with respect to ε, in the case where there is no external boundary.

Concerning the Christoffel symbols (2.13), it is not hard to check that

$$\langle \Gamma^{\varepsilon}_{\cancel{\partial\Omega},\vartheta}, p^\varepsilon, p^\varepsilon \rangle = \varepsilon I_\varepsilon \langle \Gamma_{\cancel{\partial\Omega},\vartheta}, \hat{p}^\varepsilon, \hat{p}^\varepsilon \rangle, \tag{2.32}$$

and concerning the force term (2.14) that

$$F^{\varepsilon}_{\cancel{\partial\Omega},\vartheta}(p^\varepsilon) = I_\varepsilon F_{\cancel{\partial\Omega},\vartheta}(\hat{p}^\varepsilon). \tag{2.33}$$

3. Scheme of proof of the main results: Theorem 2 and Theorem 3

In this section we give the scheme of the proof of Theorem 2 (Case (i)) and of Theorem 3 (Case (ii)). The proof is split into six parts.

3.1. **ODE Formulation.**

The first step of the proof consists in establishing the reformulation of the system in terms of an ordinary differential equation given by Theorem 1. Once this is obtained (see Section 9), we will prove in addition that the Christoffel symbols can be split into two parts: one taking into account the effect of the solid rotation and the other part encoding the effect of the exterior boundary.

First, we let:

$$\langle \Gamma_{\mathcal{S}}(q), p, p \rangle := - \begin{pmatrix} 0 \\ P_a \end{pmatrix} \times p - \omega M_a(q) \begin{pmatrix} 0 \\ \ell^\perp \end{pmatrix} \in \mathbb{R}^3. \tag{3.1}$$

We can notice that one also has

$$\langle \Gamma_{\mathcal{S}}(q), p, p \rangle = - \begin{pmatrix} 0 \\ P \end{pmatrix} \times p - \omega M(q) \begin{pmatrix} 0 \\ \ell^\perp \end{pmatrix} \in \mathbb{R}^3,$$

since the extra terms cancel out. Let us recall that P_a and P are defined in (1.10).

Next, for every $j, k, l \in \{1, 2, 3\}$, we set

$$(\Gamma_{\partial\Omega})^j_{k,l}(q) := \frac{1}{2} \int_{\partial\Omega} \left(\frac{\partial \varphi_j}{\partial \tau} \frac{\partial \varphi_k}{\partial \tau} K_l + \frac{\partial \varphi_j}{\partial \tau} \frac{\partial \varphi_l}{\partial \tau} K_k - \frac{\partial \varphi_k}{\partial \tau} \frac{\partial \varphi_l}{\partial \tau} K_j \right) (q, \cdot) \, \mathrm{d}s, \tag{3.2}$$

and we associate correspondingly $\Gamma_{\partial\Omega}(q) \in \mathcal{BL}(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R}^3)$ so that for $p = (p_1, p_2, p_3) \in \mathbb{R}^3$:

$$\langle \Gamma_{\partial\Omega}(q), p, p \rangle := \left(\sum_{1 \leqslant k,l \leqslant 3} (\Gamma_{\partial\Omega})^j_{k,l}(q) \, p_k p_l \right)_{1 \leqslant j \leqslant 3} \in \mathbb{R}^3. \tag{3.3}$$

The Christoffel symbols satisfy the following relation.

Proposition 2. *For every $q \in \mathcal{Q}$ and for every $p \in \mathbb{R}^3$ we have:*

$$\langle \Gamma(q), p, p \rangle = \langle \Gamma_{\mathcal{S}}(q), p, p \rangle + \langle \Gamma_{\partial\Omega}(q), p, p \rangle. \tag{3.4}$$

The proof of Proposition 2 is given in Section 9.4. We emphasize that in (3.4) and the expressions above, with respect to (1.11c), there is no derivative with respect to q, that is, no shape derivative. We will see that in this decomposition, $\Gamma_{\partial\Omega}$ obeys a softer scaling law with respect to ε than Γ (compare (2.32) and (6.9) below.)

Next we work on Equations (1.18) with a shrunk solid, that is

$$(q^\varepsilon)' = p^\varepsilon, \tag{3.5a}$$

$$M^\varepsilon(q^\varepsilon)(p^\varepsilon)' + \langle \Gamma^\varepsilon(q^\varepsilon), p^\varepsilon, p^\varepsilon\rangle = F^\varepsilon(q^\varepsilon, p^\varepsilon), \tag{3.5b}$$

where

$$M^\varepsilon := M[\mathcal{S}_0^\varepsilon, m^\varepsilon, \mathcal{J}^\varepsilon, \Omega],\ \Gamma^\varepsilon := \Gamma[\mathcal{S}_0^\varepsilon, \Omega] \text{ and } F^\varepsilon := F[\mathcal{S}_0^\varepsilon, \gamma, \Omega].$$

Recall that the difference between Case (i) and Case (ii) is that $m^\varepsilon, \mathcal{J}^\varepsilon$ are given by (1.21) in the first case whereas they are given by (1.27) in the second one. The functions $M^\varepsilon(q)$, $\langle \Gamma^\varepsilon(q), p, p\rangle$ and $F^\varepsilon(q,p)$ are defined for q in $\mathcal{Q}^\varepsilon$ and for p in $\mathbb{R}^3$.

3.2. **Behaviour of the energy as $\varepsilon \to 0^+$.** We will of course need uniform estimates of p^ε as $\varepsilon \to 0^+$ in order to establish the result. The energy is the natural candidate to yield such estimates. Hence we are led to consider the behavior of the energy with respect to ε. We index the energy as follows:

$$\mathcal{E}^\varepsilon(q^\varepsilon, p^\varepsilon) := \frac{1}{2} M^\varepsilon(q^\varepsilon) p^\varepsilon \cdot p^\varepsilon + U^\varepsilon(q^\varepsilon), \tag{3.6}$$

where the potential energy U^ε is given by

$$U^\varepsilon(q) := -\frac{1}{2}\gamma^2 C^\varepsilon(q). \tag{3.7}$$

• *Potential energy.* Let us start with the potential energy which does not depend on whether we consider Case (i) or Case (ii). The following result establishes that the potential energy $U^\varepsilon(q^\varepsilon)$ diverges logarithmically with ε. The expansion is uniform, in the sense that the reminder is uniformly bounded, as long as the solid stays at a positive distance from the external boundary.

Lemma 3. *For any $\delta > 0$, there exists $\varepsilon_0 \in (0,1)$ and a function $U_r \in L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0}; \mathbb{R})$ such that for any (ε, q) in $\mathfrak{Q}_{\delta,\varepsilon_0}$,*

$$U^\varepsilon(q) = \frac{1}{2}\gamma^2\big(G(\varepsilon) - C_{\partial\Omega}\big) - \gamma^2 \psi_\Omega(h) + \varepsilon U_r(\varepsilon, q). \tag{3.8}$$

Lemma 3 follows from Lemma 6 below.

Although the first term in the right hand side of (3.8) diverges as ε goes to 0 it can be discarded from the energy (3.6) since it does not depend on the solid position and velocity. Hence the renormalized energy $\check{\mathcal{E}}^\varepsilon(q^\varepsilon, p^\varepsilon)$ defined by

$$\check{\mathcal{E}}^\varepsilon(q^\varepsilon, p^\varepsilon) := \frac{1}{2} M^\varepsilon(q^\varepsilon) p^\varepsilon \cdot p^\varepsilon - \gamma^2 \psi_\Omega(h^\varepsilon) + \varepsilon U_r(\varepsilon, q^\varepsilon), \tag{3.9}$$

is also conserved according to the following result which is obtained by combining (3.6) and (3.8).

Corollary 2. *Let $(q^\varepsilon, p^\varepsilon)$ and T^ε as in Theorem 2 or as in Theorem 3. Then, till T^ε, there holds:*

$$\frac{d}{dt}\check{\mathcal{E}}^\varepsilon(q^\varepsilon, p^\varepsilon) = 0. \tag{3.10}$$

• *Kinetic energy.* Let us now deal with the kinetic energy $\frac{1}{2}M^\varepsilon(q^\varepsilon)p^\varepsilon\cdot p^\varepsilon$. Let, for any $\vartheta\in\mathbb{R}$, for any $\varepsilon\in(0,1)$,

$$M_\vartheta(\varepsilon):=\left\{\begin{array}{lll} M_g^1+\varepsilon^{2-\alpha}\,M_{a,\cancel{\partial\Omega},\vartheta} & \text{if} & \alpha\leqslant 2,\\ M_{a,\cancel{\partial\Omega},\vartheta}+\varepsilon^{\alpha-2}\,M_g^1 & \text{if} & \alpha>2,\end{array}\right. \tag{3.11}$$

and, for any $p\in\mathbb{R}^3$,

$$\mathcal{E}_\vartheta(\varepsilon,p):=\frac{1}{2}M_\vartheta(\varepsilon)p\cdot p. \tag{3.12}$$

Then we have the following result.

Lemma 4. *Let $\delta>0$. In Case (i) ($\alpha=0$) and in Case (ii) ($\alpha>0$), there exists $\varepsilon_0\in(0,1)$ and a function $M_r\in L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0};\mathbb{R}^{3\times3})$ depending on $\mathcal{S}_0$ and Ω, such that, for all $(\varepsilon,q)\in\mathfrak{Q}_{\delta,\varepsilon_0}$, for all $p\in\mathbb{R}^3$,*

$$\frac{1}{2}M^\varepsilon(q)p\cdot p=\varepsilon^{\min(2,\alpha)}\,\mathcal{E}_\vartheta(\varepsilon,\hat{p})+\frac{1}{2}\varepsilon^4M_r(\varepsilon,q)\hat{p}\cdot\hat{p},\ \textit{with } \hat{p}=I_\varepsilon p.$$

Lemma 4 is an immediate consequence of Proposition 4 below.

Now we slightly modify $\mathcal{E}_\vartheta$ in order to give general statements of the results covering the particular case where $\mathcal{S}_0$ is a disk. Indeed if $\mathcal{S}_0$ is not a disk, whatever ϑ, $M_{a,\cancel{\partial\Omega},\vartheta}$ is positive definite, whereas if $\mathcal{S}_0$ is a disk, for any $\vartheta\in\mathbb{R}$, the matrix $M_{a,\cancel{\partial\Omega},\vartheta}$ is diagonal, of the form $M_{a,\cancel{\partial\Omega},\vartheta}=\operatorname{diag}(0,m_{a,\cancel{\partial\Omega}},m_{a,\cancel{\partial\Omega}})$ with $m_{a,\cancel{\partial\Omega}}>0$ not depending on ϑ.

Let us define $\tilde{M}_\vartheta(\varepsilon)$, for any $\vartheta\in\mathbb{R}$, for any $\varepsilon\in(0,1)$, by setting $\tilde{M}_\vartheta(\varepsilon):=M_\vartheta(\varepsilon)$ if $\mathcal{S}_0$ is not a disk, and $\tilde{M}_\vartheta(\varepsilon):=\operatorname{diag}(1,0,0)+M_\vartheta(\varepsilon)$ if $\mathcal{S}_0$ is a disk. Then, for any $p\in\mathbb{R}^3$, we define

$$\tilde{\mathcal{E}}_\vartheta(\varepsilon,p):=\frac{1}{2}\tilde{M}_\vartheta(\varepsilon)p\cdot p. \tag{3.13}$$

Then we straightforwardly have the following universal result.

Lemma 5. *There exists $K>0$ depending only on $\mathcal{S}_0$, m^1 and $\mathcal{J}^1$ such that, for any $(\varepsilon,\vartheta,p)$ in $(0,1)\times\mathbb{R}\times\mathbb{R}^3$,*

$$K|p|^2_{\mathbb{R}^3}\leqslant\tilde{\mathcal{E}}_\vartheta(\varepsilon,p)\leqslant K^{-1}|p|^2_{\mathbb{R}^3}.$$

Let us recall that if $\mathcal{S}_0$ is a disk, in both Cases (i) and (ii), it follows directly from (1.1e) that for any $\varepsilon\in(0,1)$, $\omega^\varepsilon=\omega_0$ as long as the solution exists. Therefore combining Corollary 2 and Lemma 4 we obtain the following result for the slightly modified renormalized energy $\breve{\mathcal{E}}^\varepsilon(q^\varepsilon,\hat{p}^\varepsilon)$ defined by

$$\breve{\mathcal{E}}^\varepsilon(q^\varepsilon,\hat{p}^\varepsilon):=\varepsilon^{\min(2,\alpha)}\,\tilde{\mathcal{E}}_\vartheta(\varepsilon,\hat{p}^\varepsilon)+\frac{1}{2}\varepsilon^4M_r(\varepsilon,q^\varepsilon)\hat{p}^\varepsilon\cdot\hat{p}^\varepsilon-\gamma^2\psi_\Omega(h^\varepsilon)+\varepsilon U_r(\varepsilon,q^\varepsilon). \tag{3.14}$$

Let us recall that $\hat{p}^\varepsilon$ is defined by (2.29).

Corollary 3. *Let $(q^\varepsilon,p^\varepsilon)$ and T^ε be as in Theorem 2 or as in Theorem 3. Then, till T^ε, there holds:*

$$\frac{d}{dt}\breve{\mathcal{E}}^\varepsilon(q^\varepsilon,\hat{p}^\varepsilon)=0. \tag{3.15}$$

Roughly speaking the two most important terms in the right hand side of (3.14) are the first and third ones which are respectively of order $O(\varepsilon^{\min(2,\alpha)}\,|\hat{p}^\varepsilon|^2_{\mathbb{R}^3})$ and $O(1)$ as long as there is no collision. Indeed the uniformity in ε in Lemma 3 and Lemma 4 combined with the conservation property stated in Corollary 3 allows to get the following counterpart of Corollary 1.

Corollary 4. *Let $(q^\varepsilon, p^\varepsilon)$ satisfies the assumptions of Theorem 2 ($\alpha = 0$) or of Theorem 3 ($\alpha > 0$). Assume that there exists $T > 0$, $\delta > 0$ and $\varepsilon_0 \in (0,1)$ such that for $\varepsilon \in (0, \varepsilon_0)$, $T^\varepsilon \geqslant T$ and on $[0,T]$, $(\varepsilon, q^\varepsilon)$ is in $\mathfrak{Q}_{\delta,\varepsilon_0}$. Then, reducing $\varepsilon_0 \in (0,1)$ if necessary, there exists $K > 0$ depending only on $\mathcal{S}_0, \Omega, p_0, \gamma, m^1, \mathcal{J}^1, \delta$ such that for $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon^{\min(1,\frac{\alpha}{2})} |\hat{p}^\varepsilon|_{\mathbb{R}^3} \leqslant K$ on $[0,T]$.*

The proof of Corollary 4 is given in Section 8.1.

3.3. **A drift term in the velocity of the center of mass.**

In Case (ii), Corollary 4 does not provide a uniform bound of the solid velocity. An important part of the proof consists in finding an appropriate substitute for the energy $\breve{\mathcal{E}}^\varepsilon(q^\varepsilon, \hat{p}^\varepsilon)$ which allows a better control on the body velocity. This will be accomplished below by a modulated energy (see Section 3.5), which, roughly speaking, consists in applying the energy $\mathcal{E}_\vartheta$ (see (3.12)) to some modified version of p^ε, which we will denote by $\tilde{\tilde{p}}^\varepsilon$.

This modulation is driven by the leading terms of the electric-type potential. We will establish in Section 5.2.1 the following result regarding the expansion of $C^\varepsilon(q)$ with respect to ε. Let, for $q := (\vartheta, h) \in \mathbb{R} \times \Omega$,

$$\psi_c(q) := D_h \psi_\Omega(h) \cdot \zeta_\vartheta. \tag{3.16}$$

Above D_h denotes the derivative with respect to h.

Lemma 6. *Let $\delta > 0$. There exists $\varepsilon_0 \in (0,1)$ and a function $C_r \in L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0}; \mathbb{R})$ such that for any (ε, q) in $\mathfrak{Q}_{\delta,\varepsilon_0}$,*

$$C^\varepsilon(q) = -G(\varepsilon) + C_{\partial\Omega} + 2\psi_\Omega(h) + 2\varepsilon\psi_c(q) + \varepsilon^2 C_r(\varepsilon, q). \tag{3.17}$$

Observe that Lemma 3 follows from Lemma 6 by setting $U_r(\varepsilon, q) := \frac{1}{2}\big(\gamma^2 \psi_c(q) + \varepsilon C_r(\varepsilon, q)\big)$. The proof of Lemma 6 is postponed to Section 5.3.

Now, in the same way as we defined the Kirchhoff-Routh velocity u_Ω by $u_\Omega = \nabla^\perp \psi_\Omega$ we introduce the corrector velocity u_c by

$$u_c(q) := \nabla_h^\perp \psi_c(q). \tag{3.18}$$

Observe that the function u_c depends on Ω, $\mathcal{S}_0$, ϑ and on h, whereas u_Ω depends only on Ω and h.

The modulation will consist in considering the unknown

$$\tilde{\tilde{\ell}} := \ell - \gamma(u_\Omega(h) + \varepsilon u_c(q)) \tag{3.19}$$

rather than ℓ. Let us observe that $\gamma(u_\Omega(h) + \varepsilon u_c(q))$ is the beginning of the expansion of $-\frac{1}{\gamma}\nabla_h^\perp U^\varepsilon(q)$ where $U^\varepsilon(q)$ is the electric-type potential energy defined in (3.7).

Moreover, as long as the solid does not touch the boundary, a bound of (3.19) is equivalent to a bound of ℓ. Indeed the following lemma is a direct consequence of Lemma 1 and of the definitions of u_Ω and u_c.

Lemma 7. *There exists $\delta > 0$, $\varepsilon_0 \in (0,1)$ and $K > 0$ such that for any (ε, q) in $\mathfrak{Q}_{\delta,\varepsilon_0}$ with $q = (\vartheta, h)$, $|u_\Omega(h) + \varepsilon u_c(q)|_{\mathbb{R}^3} \leqslant K$.*

3.4. **Geodesic-gyroscopic normal forms.**

We will establish that (3.5) can be put into a normal form whose structure looks like (2.6). We first introduce two definitions.

Definition 2. *We say that a vector field $F \in L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0} \times \mathbb{R}^3; \mathbb{R}^3)$ is weakly nonlinear if for any $\delta > 0$ there exists $\varepsilon_0 \in (0,1)$ depending on $\mathcal{S}_0$, m, $\mathcal{J}$, γ, Ω and δ such that for any $(\varepsilon, q, p) \in \mathfrak{Q}_{\delta,\varepsilon_0} \times \mathbb{R}^3$,*

$$|F(\varepsilon, q, p)|_{\mathbb{R}^3} \leqslant K(1 + |p|_{\mathbb{R}^3} + \varepsilon |p|^2_{\mathbb{R}^3}). \tag{3.20}$$

Definition 3. *We say that a vector field $F \in C^\infty(\mathbb{R}\times\Omega;\mathbb{R}^3)$ is weakly gyroscopic if for any $\delta > 0$ there exists $K > 0$ depending on $\mathcal{S}_0$, Ω, γ and δ such that for any smooth curve $q(t) = (\vartheta(t), h(t))$ in $\mathbb{R}\times\Omega_\delta$, we have, for any $t \geqslant 0$,*

$$\Big| \int_0^t \tilde{\tilde{p}} \cdot F(q) \Big| \leqslant \varepsilon K \big(1 + t + \int_0^t |\tilde{\tilde{p}}|^2_{\mathbb{R}^3}\big), \tag{3.21}$$

with $p = q'(t) = (\omega, \ell)$, $\tilde{\tilde{p}} = (\hat{\omega}, \tilde{\tilde{\ell}})$, where $\hat{\omega} = \varepsilon\omega$, and $\tilde{\tilde{\ell}} = \ell - \gamma(u_\Omega(h) + \varepsilon u_c(q))$.

Above $u_c(q)$ denotes the corrector velocity defined in (3.18).

We introduce

$$\hat{\omega}^\varepsilon := \varepsilon\omega^\varepsilon, \quad \tilde{\ell}^\varepsilon := \ell^\varepsilon - \gamma u_\Omega(h^\varepsilon), \quad \tilde{p}^\varepsilon := (\hat{\omega}^\varepsilon, \tilde{\ell}^\varepsilon),$$
$$\tilde{\tilde{\ell}}^\varepsilon := \ell^\varepsilon - \gamma(u_\Omega(h^\varepsilon) + \varepsilon u_c(q^\varepsilon)) \text{ and } \tilde{\tilde{p}}^\varepsilon := (\hat{\omega}^\varepsilon, \tilde{\tilde{\ell}}^\varepsilon).$$

Recall also that $\hat{p}^\varepsilon$ was defined in (2.29). The normal forms are as follows.

Proposition 3. *Let $\delta > 0$. There exists $\varepsilon_0 \in (0,1)$ and*

- *$H_r, \tilde{H}_r \in L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0} \times \mathbb{R}^3;\mathbb{R}^3)$ depending on $\mathcal{S}_0$, γ and Ω, weakly nonlinear in the sense of Definition 2;*
- *$\mathsf{E}^1_b \in C^\infty(\mathbb{R}\times\Omega;\mathbb{R}^3)$ depending on $\mathcal{S}_0$ and Ω, weakly gyroscopic in the sense of Definition 3;*

such that Equation (3.5) can be recast as

$$M^1_g(\hat{p}^\varepsilon)' = F_{\partial\Omega,\vartheta^\varepsilon}(\tilde{p}^\varepsilon) + \varepsilon H_r(\varepsilon, q^\varepsilon, \hat{p}^\varepsilon), \tag{3.22}$$

in Case (i), and

$$\varepsilon^{\min(2,\alpha)}\, \tilde{M}_{\vartheta^\varepsilon}(\varepsilon)\big(\tilde{\tilde{p}}^\varepsilon\big)' + \varepsilon\langle\Gamma_{\partial\Omega,\vartheta^\varepsilon}, \tilde{\tilde{p}}^\varepsilon, \tilde{\tilde{p}}^\varepsilon\rangle = F_{\partial\Omega,\vartheta^\varepsilon}(\tilde{\tilde{p}}^\varepsilon) + \varepsilon\gamma^2 \mathsf{E}^1_b(q^\varepsilon) + \varepsilon^{\min(2,\alpha)}\tilde{H}_r(\varepsilon, q^\varepsilon, \tilde{\tilde{p}}^\varepsilon), \tag{3.23}$$

in Case (ii).

- **Motivations**. The normal form (3.22) will be useful in order to pass to the limit in Case (i) and the normal form (3.23) both in order to get a uniform bound of the velocity and to pass to the limit in Case (ii). It would be actually possible to deal with the case where α is small with a less accurate normal form and still get an energy estimate.

  In particular in order to get a uniform bound of the velocity in Case (ii) we will perform an estimate of an energy adapted to the normal form (3.23). Observe that should the right hand side vanish the normal form (3.23) would be the geodesic equation associated with the metric $\tilde{M}_\vartheta(\varepsilon)$. On the other hand the right hand side is the sum of terms with a quite remarkable structure: the leading term $F_{\partial\Omega,\vartheta^\varepsilon}(\tilde{\tilde{p}}^\varepsilon)$ is gyroscopic in the sense of Definition 1, the electric-type term $\mathsf{E}^1_b(q^\varepsilon)$ is weakly gyroscopic in the sense of Definition 3; and the reminder $\tilde{H}_r$ is weakly nonlinear.

  Regarding the passage to the limit we will face an extra difficulty: the force, including the leading term $F_{\partial\Omega,\vartheta^\varepsilon}$, depends on the unknown $\varepsilon\vartheta^\varepsilon$ through ϑ^ε, that is singularly. This difficulty will be overcome by using some averaging effect; see Lemma 31 and (8.16) in Case (i), and (8.20) in Case (ii).

- **Ideas of the proof of Proposition 3**. To get Proposition 3, we will perform expansions of the inertia matrix, of the Christoffel symbols and of the force terms with respect to ε. Roughly speaking the leading terms coming from the force terms will be gathered into the first term of the left hand side of (3.22) and (3.23), see (7.64). A striking and crucial phenomenon is that some subprincipal contributions (that is, of order ε) of the force terms will be gathered with

the leading part of the Christoffel symbols into the second term of the left hand side of (3.23), see Lemma 30 and (7.68). The leading part of the contribution coming from the Christoffel symbols will be provided by the $\Gamma_{\mathcal{S}}$-part of the decomposition (3.4).

Remark 4. *The normal forms above are inspired by the case without external boundary (see equation* (2.6)*) and by the paper* [2] *where the authors consider the motion of a light charged particle in a slowly varying electromagnetic field. The equation of motion for the particle is an ordinary differential equation involving a small parameter in front of the higher order term. In order to restore some uniformity with respect to the small parameter they use a modulation, subtracting to the particle velocity the $|B|^{-2}E\times B$ drift, and a normal form, see* [2, Eq. (3.5)], *where the only remaining singular term appears through a Lorentz gyroscopic force. This allows to tackle the convergence of the particle motion to the so-called guiding center motion despite the fast oscillations induced by the gyroscopic force.*

However our drift term $(0,\gamma(u_\Omega(h)+\varepsilon u_c(q)))$ does not enter this framework. Actually the use of the $|B|^{-2}E\times B$ drift could give a modulated energy estimate only in the case $\alpha\leqslant 1$, and in particular not in the case of a solid with a fixed homogeneous density ($\alpha=2$). Moreover it would not be adapted to the passage to the limit.

3.5. **Modulated energy estimates.** In Case (i), Corollary 4 provides a uniform bound of ℓ^ε as long as the body stays at a positive distance from the external boundary. As mentioned above, in Case (ii), Corollary 4 fails to provide such a bound.

However the structure established in Proposition 3 will allow us to obtain an estimate of the modulated energy

$$\tilde{\mathcal{E}}_\vartheta(\varepsilon,\tilde{\tilde{p}}^\varepsilon), \tag{3.24}$$

where the functional $\tilde{\mathcal{E}}_\vartheta$ is defined in (3.13). Since the equation (3.23) looks like the equation (2.6) of the case without external boundary for which the total energy is the kinetic energy defined in (2.23) alone, one may hope to have a good behaviour of the the modulated energy (3.24) when time proceeds. Indeed we have the following result.

Lemma 8. *Let $(q^\varepsilon,p^\varepsilon)$ satisfies the assumptions of Theorem 3. Then, as long as the solution exists,*

$$\frac{d}{dt}\tilde{\mathcal{E}}_\vartheta(\varepsilon,\tilde{\tilde{p}}^\varepsilon)=\varepsilon^{\max(1-\alpha,-1)}\,\gamma^2\tilde{\tilde{p}}^\varepsilon\cdot\mathsf{E}^1_b(q^\varepsilon)+\tilde{\tilde{p}}^\varepsilon\cdot\tilde{H}_r(\varepsilon,q,\tilde{\tilde{p}}^\varepsilon). \tag{3.25}$$

Lemma 8 is proved in Section 8.2.

Then we use that Corollary 4 and Lemma 7 already give us that $\varepsilon\tilde{\tilde{p}}^\varepsilon$ is bounded, and then that $\tilde{H}_r\in L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0}\times\mathbb{R}^3;\mathbb{R}^3)$ is weakly nonlinear in the sense of Definition 2, that E^1_b is weakly gyroscopic in the sense of Definition 3 (using Lemma 1), Lemma 5 and Gronwall's lemma to get the following result.

Corollary 5. *Let $(q^\varepsilon,p^\varepsilon)$ satisfies the assumptions of Theorem 3. Assume that there exists $T>0$, $\delta>0$ and $\varepsilon_0\in(0,1)$ such that for $\varepsilon\in(0,\varepsilon_0)$, $T^\varepsilon\geqslant T$ and on $[0,T]$, $(\varepsilon,q^\varepsilon)$ is in $\mathfrak{Q}_{\delta,\varepsilon_0}$. Then, reducing $\varepsilon_0\in(0,1)$ if necessary, there exists $K>0$ depending only on $\mathcal{S}_0,\Omega,p_0,\gamma,m^1,\mathcal{J}^1,\delta$ and T such that for $\varepsilon\in(0,\varepsilon_0)$, $|\hat{p}^\varepsilon|_{\mathbb{R}^3}\leqslant K$ on $[0,T]$.*

Corollary 5 therefore provides the same estimates for Case (ii) than Corollary 4 for Case (i).

3.6. **Passage to the limit.** We deduce from Corollary 5 two different results. The first result regards the lifetime T^ε of $(q^\varepsilon, p^\varepsilon)$ of the solution, which can be only limited by a possible encounter between the solid and the boundary $\partial\Omega$.

Lemma 9. *There exists $\varepsilon_0 > 0$, $\overline{T} > 0$ and $\overline{\delta} > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, we have*

$$T^\varepsilon \geqslant \overline{T} \quad \text{and on } [0, \overline{T}], \quad (\varepsilon, q^\varepsilon) \in \mathfrak{Q}_{\overline{\delta}, \varepsilon_0}. \tag{3.26}$$

The proof of Lemma 9 is given in Section 8.3.

The second result establishes the desired convergence on any time interval during which we have a minimal distance between $\mathcal{S}^\varepsilon(q)$ and $\partial\Omega$, uniform for small ε.

Lemma 10. *Let $\varepsilon_1 > 0$, $\check{\delta} > 0$ and $\check{T} > 0$ with $\check{T} < T_{(i)}$ in Case (i), and suppose that for any $\varepsilon \in (0, \varepsilon_1)$, we have*

$$(\varepsilon, q^\varepsilon) \in \mathfrak{Q}_{\check{\delta}, \varepsilon_1} \quad \text{on } [0, \check{T}]. \tag{3.27}$$

Then

- *in Case (i), $(h^\varepsilon, \varepsilon\vartheta^\varepsilon) \longrightarrow (h_{(i)}, 0)$ in $W^{2,\infty}([0, \check{T}]; \mathbb{R}^3)$ weak-$\star$;*
- *in Case (ii), $h^\varepsilon \longrightarrow h_{(ii)}$ in $W^{1,\infty}([0, \check{T}]; \mathbb{R}^2)$ weak-$\star$.*

Let us recall that $(h_{(i)}, T_{(i)})$ denotes the maximal solution of (1.22) and $h_{(ii)}$ the global solution of (1.28). The proof of Lemma 10 is given in Section 8.4. It consists in passing to the weak limit, with the help of all a priori bounds, in each term of (3.22) or (3.23).

Then to get the precise results of Theorems 2 and 3, it will only remain to extend the time interval on which the above convergences are valid. This is done in Section 8.

Organization of the rest of the paper. Now the rest of the paper is organized as follows. In Section 4, we prove the normal forms (3.22) and (3.23), relying on propositions on the asymptotic expansions of the inertia matrix and the force term as ε goes to 0. These propositions are respectively proved in Sections 6 and 7, relying on lemmas concerning the asymptotic expansions of stream functions, proved in Section 5. Next in Section 8, we prove the results about the renormalized and modulated energy estimates (Corollary 4 and Lemma 8), and concerning the passage to the limit (including Lemmas 9 and 10) in order to conclude the proofs of Theorems 2 and 3. Finally Theorem 1, Proposition 1 and Proposition 2, which are independent of ε, are proved in Section 9.

4. Conditional proof of the normal forms

The proof of the normal forms (3.22)-(3.23) consists first in expanding the functions $M^\varepsilon(q)$, $\langle \Gamma^\varepsilon(q), p, p\rangle$ and $F^\varepsilon(q, p)$ with respect to ε and then in plugging these expansions into (3.5). Next, further modifications are in order to reach the exact forms (3.22) and (3.23).

4.1. **Expansions of the inertia matrix and of the force.** We begin by giving the expansions in terms of ε of the inertia matrix and of the force.

Inertia matrix. The first part concerns the expansion of the inertia matrix $M_a^\varepsilon(q)$ which is the counterpart for the body of size ε of the added mass $M_a(q)$ defined in (1.9b).

Proposition 4. *Let $\delta > 0$. There exists $\varepsilon_0 \in (0,1)$ and a function $M_r \in L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0};\mathbb{R}^{3\times 3})$ depending on $\mathcal{S}_0$ and Ω, such that, for all $(\varepsilon, q) \in \mathfrak{Q}_{\delta,\varepsilon_0}$,*

$$M_a^\varepsilon(q) = \varepsilon^2 I_\varepsilon \Big(M_{a,\cancel{\partial\Omega},\vartheta} + \varepsilon^2 M_r(\varepsilon, q) \Big) I_\varepsilon. \tag{4.1}$$

Let us recall that the matrix $M_{a,\cancel{\partial\Omega},\vartheta}$ is defined in (2.10) and (2.11) .
Force term. Now we turn to the force term. Indeed as hinted after Proposition 3, there is some key combinations between terms coming from the expansion of $\langle \Gamma^\varepsilon(q), p, p\rangle$ and the ones coming from $F^\varepsilon(q,p)$. We therefore introduce, for $(\varepsilon, q, p) \in \mathfrak{Q} \times \mathbb{R}^3$,

$$H^\varepsilon(q,p) := F^\varepsilon(q,p) - \langle \Gamma^\varepsilon(q), p, p\rangle, \tag{4.2}$$

and state here the expansion of $H^\varepsilon(q,p)$. Roughly speaking, Proposition 5 below establishes that the subprincipal term of $H^\varepsilon(q,p)$ can be decomposed as the sum of three terms:

- a term which can be interpreted as a Christoffel-type term,
- a term which has a special structure allowing the gain of one factor ε by integration by parts in time,
- a term which can be absorbed by the principal term, up to introducing a corrector velocity.

Let us introduce in details these two last terms, which are denoted below respectively by E_b^1 and E_c^1.

The weakly gyroscopic subprincipal term E_b^1. Let us introduce the geometrical constant 2×2 matrix

$$\sigma := \int_{\partial\mathcal{S}_0} \frac{\partial \psi^{-1}_{\cancel{\partial\Omega}}}{\partial n}(X)\, X \otimes X^\perp \,\mathrm{d}s(X) + \zeta \otimes \zeta^\perp = \int_{\partial\mathcal{S}_0} \frac{\partial \psi^{-1}_{\cancel{\partial\Omega}}}{\partial n}(X)\, (X \otimes X^\perp - \zeta\otimes\zeta^\perp) \,\mathrm{d}s(X), \tag{4.3}$$

which only depends on $\mathcal{S}_0$,

$$\sigma^s := \frac{1}{2}(\sigma + \sigma^t), \tag{4.4}$$

its symmetric part and the associated field force $\mathsf{E}_b^1(q)$ defined, for $q = (\vartheta, h)$ in $\mathbb{R}\times\Omega$, by

$$\mathsf{E}_b^1(q) := \begin{pmatrix} -\langle D_x^2 \psi^0_{\cancel{\partial\mathcal{S}}}(h,h), R(-2\vartheta)\sigma^s\rangle_{\mathbb{R}^{2\times 2}} \\ 0 \\ 0 \end{pmatrix}. \tag{4.5}$$

Lemma 11. *The vector field $\mathsf{E}_b^1 \in C^\infty(\mathbb{R}\times\Omega;\mathbb{R}^3)$ defined by (4.5) is weakly gyroscopic in the sense of Definition 3.*

Proof of Lemma 11. First, for any smooth curve $q(t) = (\vartheta(t), h(t))$ in $\mathbb{R}\times\Omega$ there holds

$$\begin{aligned} -\int_0^t \hat{p}\cdot \frac{1}{\varepsilon}\mathsf{E}_b^1(q) = {}& \frac{1}{2}\langle D_x^2\psi^0_{\cancel{\partial\mathcal{S}}}(h,h), R(-2\vartheta + \frac{\pi}{2})\sigma^s\rangle_{\mathbb{R}^{2\times 2}}(t) - \frac{1}{2}\langle D_x^2\psi^0_{\cancel{\partial\mathcal{S}}}(0,0), R(\frac{\pi}{2})\sigma^s\rangle_{\mathbb{R}^{2\times 2}} \\ & - \int_0^t \langle D_x^3\psi^0_{\cancel{\partial\mathcal{S}}}(h,h)\cdot\ell, R(-2\vartheta+\frac{\pi}{2})\sigma^s\rangle_{\mathbb{R}^{2\times 2}}\,\mathrm{d}s, \end{aligned} \tag{4.6}$$

where $p = q' = (\omega, \ell)$, $\tilde{\tilde{p}} = (\hat\omega, \tilde{\tilde{\ell}})$, $\hat\omega = \varepsilon\omega$, and $\tilde{\tilde{\ell}} = \ell - \gamma(u_\Omega(h) + \varepsilon u_c(q))$.
The conclusion then follows from crude bounds, Lemma 1, the smoothness of the function $\psi^0_{\cancel{\partial\mathcal{S}}}$ defined in (1.23) and Lemma 7. □

The drift subprincipal term E^1_c. Let us introduce the force field $\mathsf{E}^1_c(q)$ defined, for $q = (\vartheta, h)$ in $\mathbb{R} \times \Omega$, by

$$\mathsf{E}^1_c(q) := -\begin{pmatrix} \zeta_\vartheta \cdot u_c(q) \\ (u_c(q))^\perp \end{pmatrix}. \tag{4.7}$$

Above $u_c(q)$ denotes the corrector velocity defined in (3.18).

Let

$$p := (\omega, \ell) \quad \tilde{p} := (\hat{\omega}, \tilde{\ell}), \quad \hat{\omega} := \varepsilon\omega, \text{ and } \tilde{\ell} := \ell - \gamma u_\Omega(h). \tag{4.8}$$

The asymptotic expansion that we obtain for the force term is as follows.

Proposition 5. *Let $\delta > 0$. There exists $\varepsilon_0 \in (0, 1)$ and $H_r \in L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0} \times \mathbb{R}^3; \mathbb{R}^3)$ depending on $\mathcal{S}_0$, γ and Ω, weakly nonlinear in the sense of Definition 2, such that for all $(\varepsilon, q) \in \mathfrak{Q}_{\delta,\varepsilon_0}$, with $q = (\vartheta, h)$, for all $p \in \mathbb{R}^3$,*

$$H^\varepsilon(q, p) = I_\varepsilon\Big(F_{\cancel{\partial\Omega},\vartheta}(\tilde{p}) + \varepsilon \mathsf{H}^1(q, \tilde{p}) + \varepsilon^2 H_r(\varepsilon, q, \tilde{p})\Big), \tag{4.9}$$

with

$$\mathsf{H}^1(q, \tilde{p}) := -\langle \Gamma_{\cancel{\partial\Omega},\vartheta}, \tilde{p}, \tilde{p}\rangle + \gamma^2\big(\mathsf{E}^1_b(q) + \mathsf{E}^1_c(q)\big). \tag{4.10}$$

Let us recall that $F_{\cancel{\partial\Omega}}(\tilde{p})$ is defined in (2.14) and $\Gamma_{\cancel{\partial\Omega},\vartheta}$ is defined in (2.13).

The proofs of Proposition 4 and Proposition 5 rely on the asymptotic expansions of stream and potential functions with respect to ε. These expansions involve two scales corresponding respectively to variations over length $O(1)$ and $O(\varepsilon)$ respectively on $\partial\Omega$ and $\partial\mathcal{S}^\varepsilon(q)$. The profiles appearing in these expansions are obtained by successive corrections, considering alternatively at their respective scales the body boundary from which the external boundary seems far away and the external boundary from which the body seems tiny, so that good approximations are given respectively by the case without external boundary and without the body.

Then we plug these expansions into the expressions of the Christoffel symbol $\langle \Gamma^\varepsilon(q), p, p\rangle$ and of the force fields E^ε and $p \times B^\varepsilon$ and compute the leading terms of the resulting expansions. We will use a lemma due to Lamb, cf. Lemma 22, to exchange some normal and tangential components in some trilinear integrals over the body boundary. In particular, we will make appear in several terms of the expansions of E^ε and B^ε some coefficients of the added inertia of the solid as if the external boundary was not there. Strikingly this allows to combine the subprincipal terms of the expansions of E and B with the leading term of the expansion of Γ, see Lemma 30 and (7.68).

4.2. **From Propositions 4 and 5 to Proposition 3.** We focus on the more delicate Case (ii). Case (i) can be proved with the same strategy with some simplifications. In order to prove (3.23) we have to perform additional manipulations. The first one is due to the fact that the expansion of H^ε above involves $\tilde{\ell}$ rather than $\tilde{\tilde{\ell}}$. Consequently, to begin with, we modify the asymptotic expansion (4.9) of H^ε as follows, by changing the arguments of the functions in the right hand side.

Proposition 6. *Let $\delta > 0$. There exists $\varepsilon_0 \in (0, 1)$ and $\hat{H}_r \in L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0} \times \mathbb{R}^3; \mathbb{R}^3)$ depending on $\mathcal{S}_0$, γ and Ω, weakly nonlinear, such that, for all $(\varepsilon, q) \in \mathfrak{Q}_{\delta,\varepsilon_0}$, with $q = (\vartheta, h)$, for all $p := (\omega, \ell) \in \mathbb{R}^3$,*

$$H^\varepsilon(q, p) = I_\varepsilon\Big(F_{\cancel{\partial\Omega},\vartheta}(\tilde{\tilde{p}}) - \varepsilon\langle \Gamma_{\cancel{\partial\Omega},\vartheta}, \tilde{\tilde{p}}, \tilde{\tilde{p}}\rangle + \varepsilon\gamma^2 \mathsf{E}^1_b(q) + \varepsilon^2 \hat{H}_r(\varepsilon, q, \tilde{\tilde{p}})\Big), \tag{4.11}$$

with $\tilde{\tilde{p}} := (\hat{\omega}, \tilde{\tilde{\ell}})$, where $\hat{\omega} := \varepsilon\omega$, and $\tilde{\tilde{\ell}} := \ell - \gamma(u_\Omega(h) + \varepsilon u_c(q))$.

From Proposition 5 to Proposition 6. Let us take Proposition 5 for granted and let us see how to infer Proposition 6.

Considering (2.19) and (4.7), we have

$$F_{\not{\partial\Omega},\vartheta}(\tilde{\tilde{p}}) = F_{\not{\partial\Omega},\vartheta}(\tilde{p}) + \varepsilon\gamma^2 \mathsf{E}^1_c(q). \tag{4.12}$$

This relation is the reason why we introduced $\tilde{\tilde{\ell}}$: the part E^1_c of the subprincipal term H^1 can be absorbed by the principal term up to a modification of size ε of the arguments.

Combining (4.10) and (4.12) we infer that

$$\begin{aligned} &F_{\not{\partial\Omega},\vartheta}(\tilde{p}) + \varepsilon \mathsf{H}^1(q,\tilde{p}) \\ &\qquad = F_{\not{\partial\Omega},\vartheta}(\tilde{\tilde{p}}) - \varepsilon\langle \Gamma_{\not{\partial\Omega},\vartheta}, \tilde{\tilde{p}}, \tilde{\tilde{p}}\rangle + \varepsilon\gamma^2 \mathsf{E}^1_b(q) - 2\varepsilon^2\gamma\langle \Gamma_{\not{\partial\Omega},\vartheta}, \tilde{\tilde{p}}, p_c(q)\rangle - \varepsilon^3\gamma^2\langle \Gamma_{\not{\partial\Omega},\vartheta}, p_c(q), p_c(q)\rangle, \end{aligned}$$

where $p_c(q) := (0, u_c(q))$. It therefore remains to deduce from (4.9) that the function $\hat{H}_r$ defined by

$$\hat{H}_r(\varepsilon, q, \tilde{\tilde{p}}) := H_r(\varepsilon, q, \tilde{\tilde{p}} + \varepsilon\gamma p_c(q)) - 2\gamma\langle \Gamma_{\not{\partial\Omega},\vartheta}, \tilde{\tilde{p}}, p_c(q)\rangle - \varepsilon\gamma^2\langle \Gamma_{\not{\partial\Omega},\vartheta}, p_c(q), p_c(q)\rangle$$

is convenient.

Proof of Proposition 3. Using the Propositions 4 and 6, and recalling definition (2.29), the equation (3.5) can be recast as follows:

$$\begin{aligned} \Big(\varepsilon^\alpha M^1_g + \varepsilon^2 M_{a,\not{\partial\Omega},\vartheta^\varepsilon} + \varepsilon^4 M_r(\varepsilon, q^\varepsilon)\Big)(\hat{p}^\varepsilon)' = F_{\not{\partial\Omega},\vartheta^\varepsilon}(\tilde{\tilde{p}}^\varepsilon) - \varepsilon\langle \Gamma_{\not{\partial\Omega},\vartheta^\varepsilon}, \tilde{\tilde{p}}^\varepsilon, \tilde{\tilde{p}}^\varepsilon\rangle + \varepsilon\gamma^2 \mathsf{E}^1_b(q) \\ + \varepsilon^2 \hat{H}_r(\varepsilon, q^\varepsilon, \tilde{\tilde{p}}^\varepsilon). \end{aligned} \tag{4.13}$$

In the case where $\mathcal{S}_0$ is a disk, an extra difficulty is that the matrix appearing in the second term is singular. Indeed, for any $\vartheta \in \mathbb{R}$, the matrix $M_{a,\not{\partial\Omega},\vartheta}$ is diagonal, of the form $M_{a,\not{\partial\Omega},\vartheta} = \text{diag}\,(0, m_{a,\not{\partial\Omega}}, m_{a,\not{\partial\Omega}})$ with $m_{a,\not{\partial\Omega}} > 0$ not depending on ϑ. Let us regularize the matrix $M_{a,\not{\partial\Omega},\vartheta}$ into

$$\tilde{M}_{a,\not{\partial\Omega},\vartheta} := \text{diag}\,(1, m_{a,\not{\partial\Omega}}, m_{a,\not{\partial\Omega}}).$$

As mentioned in Section 1.4 in the case where $\mathcal{S}_0$ is a disk, it follows directly from (1.1e) that the rotation ϑ^ε satisfies, for any $\varepsilon \in (0,1)$, $(\vartheta^\varepsilon)''(t) = 0$ as long as the solution exists. Therefore the equation (4.13) is valid as well with $\tilde{M}_{a,\not{\partial\Omega},\vartheta^\varepsilon}$ instead of $M_{a,\not{\partial\Omega},\vartheta^\varepsilon}$ in the second term. Recalling the notation (3.11) the right hand side of (4.13) reads

$$\varepsilon^{\min(2,\alpha)}\left(\tilde{M}_{\vartheta^\varepsilon}(\varepsilon) + \varepsilon^{4-\min(2,\alpha)}\, M_r(\varepsilon, q^\varepsilon)\right)(\hat{p}^\varepsilon)' \tag{4.14}$$

whether $\mathcal{S}_0$ is a disk or not.

We need further modifications to this equation in order to achieve the forms (3.22)-(3.23), due to the fact that (4.14) contains some extra lower-order terms, and that the time derivative is applied to ℓ^ε rather than to $\tilde{\ell}^\varepsilon$.

Let us start with the first discrepancy. Since $M_r \in L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0}; \mathbb{R}^{3\times3})$, reducing $\varepsilon_0 \in (0,1)$ if necessary, we get the following. For any $(\varepsilon, q) \in \mathfrak{Q}_{\delta,\varepsilon_0}$, with $q = (\vartheta, h)$, $\tilde{M}_\vartheta(\varepsilon) + \varepsilon^{4-\min(2,\alpha)}\, M_r(\varepsilon, q)$ is invertible and $M_{r,1}, M_{r,2}$ defined, for $(\varepsilon, q) \in \mathfrak{Q}_{\delta,\varepsilon_0}$, by

$$\begin{aligned} M_{r,1}(\varepsilon, q) &:= \tilde{M}_\vartheta(\varepsilon)\Big(\tilde{M}_\vartheta(\varepsilon) + \varepsilon^{4-\min(2,\alpha)}\, M_r(\varepsilon, q)\Big)^{-1}, \\ M_{r,2}(\varepsilon, q) &:= \varepsilon^{-2}\left(M_{r,1}(\varepsilon, q) - Id\right), \end{aligned}$$

are in $L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0}; \mathbb{R}^{3\times3})$.

Let, for any $(q,\tilde{\tilde{p}},\varepsilon)\in\mathcal{Q}^\varepsilon\times\mathbb{R}^3\times[0,1)$,

$$\overline{H}_r(\varepsilon,q,\tilde{\tilde{p}}) := M_{r,1}(\varepsilon,q)\hat{H}_r(\varepsilon,q,\tilde{\tilde{p}}) + M_{r,2}(\varepsilon,q^\varepsilon)\Big(F_{\cancel{\partial\Omega},\vartheta}(\tilde{\tilde{p}}) - \varepsilon\langle\Gamma_{\cancel{\partial\Omega},\vartheta},\tilde{\tilde{p}},\tilde{\tilde{p}}\rangle + \varepsilon\gamma^2\mathsf{E}^1_b(q)\Big). \tag{4.15}$$

Using that $\hat{H}_r\in L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0}\times\mathbb{R}^3;\mathbb{R}^3)$ is weakly nonlinear in the sense of Definition 2 and that $M_{r,1}$ and $M_{r,2}$ are in $L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0};\mathbb{R}^{3\times3})$ we obtain that $\overline{H}_r$ is weakly nonlinear as well.

Now equation (4.13) can be rephrased as

$$\varepsilon^{\min(2,\alpha)}\,\tilde{M}_{\vartheta^\varepsilon}(\varepsilon)(\hat{p}^\varepsilon)' + \varepsilon\langle\Gamma_{\cancel{\partial\Omega},\vartheta^\varepsilon},\tilde{\tilde{p}}^\varepsilon,\tilde{\tilde{p}}^\varepsilon\rangle = F_{\cancel{\partial\Omega},\vartheta^\varepsilon}(\tilde{\tilde{p}}^\varepsilon) + \varepsilon\gamma^2\mathsf{E}^1_b(q) + \varepsilon^2\overline{H}_r(\varepsilon,q^\varepsilon,\tilde{\tilde{p}}^\varepsilon). \tag{4.16}$$

On the other hand, for the second discrepancy, we compute

$$(\tilde{\tilde{\ell}}^\varepsilon)' = (\ell^\varepsilon)' - H_r^\flat(\varepsilon,q^\varepsilon,\tilde{\tilde{p}}^\varepsilon), \tag{4.17}$$

where

$$\begin{aligned} H_r^\flat(\varepsilon,q,\tilde{\tilde{p}}) &= \gamma\tilde{\tilde{\ell}}\cdot(\nabla u_\Omega)(h) + \gamma^2(u_\Omega(h)+\varepsilon u_c(q))\cdot\nabla u_\Omega(h) + \gamma D_\vartheta u_c(q)\hat{\omega} \\ &\quad + \varepsilon\gamma D_h u_c(q)\cdot\big(\tilde{\tilde{\ell}} + \gamma(u_\Omega(h)+\varepsilon u_c(q))\big). \end{aligned}$$

Let $\tilde{H}_r$ be defined by

$$\varepsilon^{\min(2,\alpha)}\tilde{H}_r(\varepsilon,q,p) := \varepsilon^2\overline{H}_r(\varepsilon,q,p) - \Big(\varepsilon^\alpha M^1_g + \varepsilon^2 M_{a,\cancel{\partial\Omega},\vartheta}\Big)\begin{bmatrix}0\\ H_r^\flat(\varepsilon,q,p)\end{bmatrix}. \tag{4.18}$$

Then (3.23) is obtained by combining (4.16), (4.17) and (4.18). Moreover $\tilde{H}_r\in L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0}\times\mathbb{R}^3;\mathbb{R}^3)$ and is weakly nonlinear in the sense of Definition 2.

Hence the proof of Proposition 3 will be complete once Propositions 4 and 5 are established. □

5. Asymptotic development of the stream and potential functions

In this section, we establish asymptotic expansions for the circulation stream function and the Kirchhoff potentials in the domain $\mathcal{F}^\varepsilon(q)$, as ε tends to 0^+. The asymptotic analysis of the Laplace equation when the size of an inclusion goes to 0 has been deeply studied, cf. for example [17] and [13]. However to our knowledge the results of this section are not covered by the literature.

5.1. **A few reminders about single-layer potentials.** In order to get the asymptotic expansions hinted above, we will look for a representation of these stream and potential functions as a superposition of single-layer integrals supported by the two connected components $\partial\mathcal{S}^\varepsilon(q)$ and $\partial\Omega$ of the boundary of the fluid domain $\mathcal{F}^\varepsilon(q)$. In this subsection, we give a few reminders about single-layer potentials which we will use in the analysis. We refer for instance to [16] and [5].

Below we consider single-layer potentials of the form:

$$SL[\mathrm{p}_\mathcal{C}] := \int_\mathcal{C}\mathrm{p}_\mathcal{C}(y)G(\cdot-y)\mathrm{d}s(y), \tag{5.1}$$

where $\mathcal{C}$ is a smooth Jordan curve in the plane and $\mathrm{p}_\mathcal{C}$ belongs to the Sobolev space $H^{-\frac{1}{2}}(\mathcal{C})$. We say that $\mathcal{C}$ is the support of the single-layer potential and that $\mathrm{p}_\mathcal{C}$ is a density on $\mathcal{C}$.

• *Harmonicity and trace.* The formula (5.1) defines a function in the Sobolev space $H^1_{loc}(\mathbb{R}^2)$ so that in particular, for any $\mathrm{p}_\mathcal{C}$ in $H^{-\frac{1}{2}}(\mathcal{C})$, the trace of $SL[\mathrm{p}_\mathcal{C}]$ on $\mathcal{C}$ is well-defined as a function of the Sobolev space $H^{\frac{1}{2}}(\mathcal{C})$.

• *Jump of the derivative and density.* The density $\mathsf{p}_{\mathcal{C}}$ is equal to the jump of the normal derivative of $SL[\mathsf{p}_{\mathcal{C}}]$ across $\mathcal{C}$.

In order to state this rigorously let us be specific on the orientation of the normal. According to Jordan's theorem, the set $\mathbb{R}^2 \setminus \mathcal{C}$ has two connected components, one bounded (the interior), say $\mathcal{O}_i$, and the other one unbounded (the exterior), say $\mathcal{O}_e$. Moreover the curve $\mathcal{C}$ is the boundary of each component. We consider the restrictions of $SL[\mathsf{p}_{\mathcal{C}}]$:

$$u_i := SL[\mathsf{p}_{\mathcal{C}}]_{|\mathcal{O}_i} \text{ and } u_e := SL[\mathsf{p}_{\mathcal{C}}]_{|\mathcal{O}_e}.$$

Denote n_i (respectively n_e) the unit normal on $\mathcal{C}$ outward to $\mathcal{O}_i$ (resp. to $\mathcal{O}_e$). Then the function u_i (respectively u_e) is harmonic in $\mathcal{O}_i$ (resp. $\mathcal{O}_e$) and the traces of the normal derivatives $\frac{\partial u_i}{\partial n_i}$ and $\frac{\partial u_e}{\partial n_e}$ on each side of $\mathcal{C}$ are well-defined in $H^{-\frac{1}{2}}(\mathcal{C})$ and satisfy

$$\mathsf{p}_{\mathcal{C}} = \frac{\partial u_i}{\partial n_i} - \frac{\partial u_e}{\partial n_i}. \tag{5.2}$$

In the sequel we will make use of single-layer potentials supported on the external boundary $\partial\Omega$, on the boundary $\partial\mathcal{S}^\varepsilon(q)$ of the solid body and on the boundary $\partial\mathcal{S}_0$ of the rescaled body as well. We will not use the notations n_i nor n_e but rather the notation n which always stands for the normal outward the fluid. Hence we will have to particularly take care of the signs when referring to the formula (5.2).

• *Kernel and rank.* We will use the following facts:

$$\text{The operator } SL \text{ is Fredholm with index zero from } L^2(\mathcal{C}) \text{ to } H^1(\mathcal{C}); \tag{5.3}$$

$$\text{If } \mathsf{p}_{\mathcal{C}} \in H^{-\frac{1}{2}}(\mathcal{C}) \text{ satisfies } \int_{\mathcal{C}} \mathsf{p}_{\mathcal{C}} \, \mathrm{d}s = 0 \text{ and } SL[\mathsf{p}_{\mathcal{C}}] = 0 \text{ then } \mathsf{p}_{\mathcal{C}} = 0; \tag{5.4}$$

$$\text{If } \mathrm{Cap}(\mathcal{C}) \neq 1, \text{ then for any } \mathsf{p}_{\mathcal{C}} \in H^{-\frac{1}{2}}(\mathcal{C}), SL[\mathsf{p}_{\mathcal{C}}] = 0 \text{ implies } \mathsf{p}_{\mathcal{C}} = 0. \tag{5.5}$$

Above, with some slight abuses of notation, we omit to mention the trace operator on $\mathcal{C}$ and we write $\int_{\mathcal{C}} \mathsf{p}_{\mathcal{C}} \, \mathrm{d}s$ for the duality bracket $\langle 1, \mathsf{p}_{\mathcal{C}} \rangle_{H^{-\frac{1}{2}}(\mathcal{C}), H^{\frac{1}{2}}(\mathcal{C})}$. We refer to [16, Th. 7.17] for (5.3), to [16, Th. 8.12] for (5.4) and to [16, Th. 8.16] for (5.5).

In particular, using that $\mathrm{Cap}_{\partial\mathcal{S}_0} < 1$, we deduce the two following results.

Corollary 6. *There exists a unique smooth function* $\psi^{-1}_{\cancel{\partial\Omega}}$ *solution of* (2.16)*. Moreover,*

$$\psi^{-1}_{\cancel{\partial\Omega}} = SL[\mathsf{p}^{-1}_{\cancel{\partial\Omega}}], \text{ with } \mathsf{p}^{-1}_{\cancel{\partial\Omega}} = \frac{\partial \psi^{-1}_{\cancel{\partial\Omega}}}{\partial n}. \tag{5.6}$$

In potential theory $-\mathsf{p}^{-1}_{\cancel{\partial\Omega}}$ is called the equilibrium density of $\partial\mathcal{S}_0$.

Corollary 7. *Let g be a smooth function on $\partial\mathcal{S}_0$ such that*

$$\int_{\partial\mathcal{S}_0} g \mathsf{p}^{-1}_{\cancel{\partial\Omega}} \, \mathrm{d}s = 0. \tag{5.7}$$

Then there exists a unique bounded smooth function f such that

$$-\Delta f = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \mathcal{S}_0, \quad \text{and} \quad f = g \quad \text{on } \partial\mathcal{S}_0. \tag{5.8}$$

Moreover, there exists a unique smooth density $p_{\partial\mathcal{S}_0}$ in $C^\infty(\partial\mathcal{S}_0)$ such that $f = SL[p_{\partial\mathcal{S}_0}]$ and

$$\int_{\partial\mathcal{S}_0} p_{\partial\mathcal{S}_0} \, \mathrm{d}s = 0. \tag{5.9}$$

Finally, $f = O(|x|_{\mathbb{R}^2}^{-1})$ *at infinity and*

$$\int_{\partial \mathcal{S}_0} \frac{\partial f}{\partial n} \, \mathrm{d}s = 0. \tag{5.10}$$

Proof of Corollary 6 and 7. The uniqueness part of Corollary 6 and of Corollary 7 and the decaying at infinity in Corollary 7 can be established by considering holomorphy at infinity of appropriate functions, see for instance [5, Prop. 2.74. and Prop. 3.2.].

The existence part of Corollary 6 is given in [16, Th. 8.15]; it also follows from the properties of the single-layers potentials recalled above, in particular (5.3) and (5.5).

Regarding the existence part of Corollary 7 we proceed in two steps.

First we prove that the operator which maps $(p_{\partial \mathcal{S}_0}, C)$ in $L^2(\mathcal{S}_0) \times \mathbb{R}$ to $(SL[p_{\partial \mathcal{S}_0}] - C, \int_{\partial \mathcal{S}_0} p_{\partial \mathcal{S}_0} \, \mathrm{d}s)$ in $H^1(\mathcal{S}_0) \times \mathbb{R}$ is invertible. In order to prove this, we observe that this operator is Fredholm with index zero as a consequence of (5.3). Moreover if $(p_{\partial \mathcal{S}_0}, C)$ is in the Kernel of this operator, then $SL[p_{\partial \mathcal{S}_0} - \frac{C}{C_{\partial\Omega}} \mathsf{p}_{\partial\Omega}^{-1}] = 0$, so that according to (5.4), $p_{\partial \mathcal{S}_0} = \frac{C}{C_{\partial\Omega}} \mathsf{p}_{\partial\Omega}^{-1}$. Then using that $\int_{\partial \mathcal{S}_0} p_{\partial \mathcal{S}_0} \, \mathrm{d}s = 0$ and

$$\int_{\partial \mathcal{S}_0} \mathsf{p}_{\partial\Omega}^{-1} \, \mathrm{d}s = -1, \tag{5.11}$$

as a consequence of (2.16e) and of the second identity of (5.6), we get that $C = 0$ and therefore $p_{\partial \mathcal{S}_0} = 0$ as well.

Then $(g, 0)$ is in the image of this operator, that is there exists $(p_{\partial \mathcal{S}_0}, C) \in L^2(\mathcal{S}_0) \times \mathbb{R}$ such that

$$SL[p_{\partial \mathcal{S}_0}] - C = g \text{ on } \partial \mathcal{S}_0, \tag{5.12}$$

and (5.9). Observing that the trace of the operator SL on $\partial \mathcal{S}_0$ is self-adjoint we infer that

$$\int_{\partial \mathcal{S}_0} SL[p_{\partial \mathcal{S}_0}] \mathsf{p}_{\partial\Omega}^{-1} \, \mathrm{d}s = \int_{\partial \mathcal{S}_0} p_{\partial \mathcal{S}_0} SL[\mathsf{p}_{\partial\Omega}^{-1}] \, \mathrm{d}s = C_{\partial\Omega} \int_{\partial \mathcal{S}_0} p_{\partial \mathcal{S}_0} \, \mathrm{d}s = 0. \tag{5.13}$$

Combining (5.7), (5.11), (5.12) and (5.13) we infer that $C = 0$.

Finally the smoothness part of Corollary 6 and of Corollary 7 follows from [16, Th. 7.16] and (5.10) follows from (5.9), (5.2) and the vanishing by integration by parts of the interior contribution. ☐

• *Regular integral operators.* Since we consider single-layer potentials supported on two disjoint curves and their values on both curves, we will also be led to consider regular integral operators. We recall below some straightforward results which are useful in the sequel. Given $\mathcal{C}$ a smooth Jordan curve in $\overline{\Omega}$, we introduce, for $\delta > 0$,

$$\mathcal{C}^\delta := \{x \in \overline{\Omega} \ / \ \mathrm{dist}(x, \mathcal{C}) < \delta\},$$

and define

$$F_\delta : C^1\big(\overline{\Omega} \setminus \mathcal{C}^\delta; H^{\frac{1}{2}}(\mathcal{C})\big) \times H^{-\frac{1}{2}}(\mathcal{C}) \to C^1(\overline{\Omega} \setminus \mathcal{C}^\delta; \mathbb{R}),$$

by setting, for any $(b, \mathsf{p}_\mathcal{C}) \in C^1\big(\overline{\Omega} \setminus \mathcal{C}^\delta; H^{\frac{1}{2}}(\mathcal{C})\big) \times H^{-\frac{1}{2}}(\mathcal{C})$, for any $x \in \overline{\Omega} \setminus \mathcal{C}^\delta$,

$$F[b, \mathsf{p}_\mathcal{C}](x) := \int_\mathcal{C} b(x, y) \mathsf{p}_\mathcal{C}(y) \mathrm{d}s(y).$$

This will be applied to b defined in a larger set but singular for $x = y$; this motivates our framework for b.

Next, given another smooth Jordan curve $\tilde{\mathcal{C}}$ in $\overline{\Omega} \setminus \mathcal{C}^\delta$ and for $b \in C^1\big(\overline{\Omega} \setminus \mathcal{C}^\delta; H^{\frac{1}{2}}(\mathcal{C})\big)$, we define the operator

$$F_{\delta,b} : L^2(\mathcal{C}) \to H^1(\tilde{\mathcal{C}})$$

by setting $F_{\delta,b}(\mathsf{p}_\mathcal{C})$ as the trace of $F_\delta[b, \mathsf{p}_\mathcal{C}]$ on $\tilde{\mathcal{C}}$. We will make use of the following lemma.

Lemma 12. *Let $\delta > 0$.*

(i) The operator F_δ is bilinear continuous with a norm less than 1, in other words: for any $(b, \mathsf{p}_\mathcal{C})$ in $C^1\big(\overline{\Omega} \setminus \mathcal{C}^\delta; H^{\frac{1}{2}}(\mathcal{C})\big) \times H^{-\frac{1}{2}}(\mathcal{C})$, one has

$$\|F_\delta[b, \mathsf{p}_\mathcal{C}]\|_{C^1(\overline{\Omega}\setminus\mathcal{C}^\delta)} \leqslant \|b\|_{C^1\left(\overline{\Omega}\setminus\mathcal{C}^\delta; H^{\frac{1}{2}}(\mathcal{C})\right)} \|\mathsf{p}_\mathcal{C}\|_{H^{-\frac{1}{2}}(\mathcal{C})}.$$

(ii) If $\tilde{\mathcal{C}}$ is a smooth Jordan curve in $\overline{\Omega} \setminus \mathcal{C}^\delta$ and $b \in C^1\big(\overline{\Omega} \setminus \mathcal{C}^\delta; H^{\frac{1}{2}}(\mathcal{C})\big)$, the operator $F_{\delta,b}$ is compact from $L^2(\mathcal{C})$ to $H^1(\tilde{\mathcal{C}})$.

The proof of Lemma 12 is elementary and left to the reader.

5.2. **Statements of the results.**

5.2.1. *Circulation part.* Let $(\varepsilon, q) \in \mathfrak{Q}$. Let us recall that the function $\psi^\varepsilon(q, \cdot)$ in $\mathcal{F}^\varepsilon(q)$ is defined as the solution to the Dirichlet boundary value problem:

$$-\Delta\psi^\varepsilon(q, \cdot) = 0 \quad \text{in } \mathcal{F}^\varepsilon(q), \tag{5.14a}$$

$$\psi^\varepsilon(q, \cdot) = C^\varepsilon(q) \quad \text{on } \partial\mathcal{S}^\varepsilon(q), \tag{5.14b}$$

$$\psi^\varepsilon(q, \cdot) = 0 \quad \text{on } \partial\Omega, \tag{5.14c}$$

where the constant $C^\varepsilon(q)$ is such that:

$$\int_{\partial\mathcal{S}^\varepsilon(q)} \frac{\partial\psi^\varepsilon}{\partial n}(q, \cdot)\mathrm{d}s = -1. \tag{5.14d}$$

Here, n stands for the unit normal vector to $\partial\mathcal{S}^\varepsilon(q) \cup \partial\Omega$ directed toward the exterior of $\mathcal{F}^\varepsilon(q)$. The function ψ^ε is the counterpart, for the case where the size of the solid is of order ε, of the function ψ defined in (1.12) in the case where the size of the solid is of order 1. For any $q \in \mathfrak{Q}$, the existence and uniqueness of a solution $\psi^\varepsilon(q, \cdot)$ of (5.14) is classical.

In order to state a result establishing an asymptotic expansion of $C^\varepsilon(q)$ and of $\frac{\partial\psi^\varepsilon}{\partial n}(q, \cdot)$ on $\partial\mathcal{S}^\varepsilon(q)$ as $\varepsilon \to 0$ we need to introduce a few notations.

- *Definition of $\psi^0_{\cancel{\partial\Omega}}(q, \cdot)$ and of $P^0(q, X)$.* We denote, for any $q := (\vartheta, h)$ in $\mathbb{R} \times \Omega$, by $P^0(q, X)$ the harmonic polynomial

$$P^0(q, X) := u_\Omega(h)^\perp \cdot (R(\vartheta)X - \zeta_\vartheta). \tag{5.15}$$

Let us recall that ζ_ϑ is defined in (2.15) in term of ζ defined in (2.17).

Recalling (5.11) and the second identity of (5.6) we observe that $P^0(q, X)$ satisfies

$$\int_{\partial\mathcal{S}_0} P^0(q, \cdot)\mathsf{p}^{-1}_{\cancel{\partial\Omega}}\, \mathrm{d}s = 0. \tag{5.16}$$

Therefore, according to Corollary 7 there exists a unique smooth function $\psi^0_{\cancel{\partial\Omega}}(q, \cdot)$ satisfying

$$-\Delta\psi^0_{\cancel{\partial\Omega}}(q, \cdot) = 0 \quad \text{in } \mathbb{R}^2 \setminus \mathcal{S}_0, \tag{5.17a}$$

$$\psi^0_{\cancel{\partial\Omega}}(q, \cdot) = P^0(q, \cdot) \quad \text{on } \partial\mathcal{S}_0, \tag{5.17b}$$

and vanishing at infinity. Moreover

$$\int_{\partial\mathcal{S}_0} \frac{\partial\psi^0_{\cancel{\partial\Omega}}}{\partial n}\, \mathrm{d}s = 0. \tag{5.17c}$$

- *Definition of* $\psi^1_{\cancel{\partial\mathcal{S}}}(q,\cdot)$. We also introduce the solution $\psi^1_{\cancel{\partial\mathcal{S}}}(q,\cdot)$ of

$$-\Delta\psi^1_{\cancel{\partial\mathcal{S}}}(q,\cdot) = 0 \quad \text{in } \Omega, \tag{5.18a}$$

$$\psi^1_{\cancel{\partial\mathcal{S}}}(q,\cdot) = -(\nabla G)(\cdot - h)\cdot\zeta_\vartheta \quad \text{on } \partial\Omega. \tag{5.18b}$$

Above G denotes the Newtonian potential defined in (1.24).

The function $\psi^1_{\cancel{\partial\mathcal{S}}}$ can be expressed thanks to the function $\psi^0_{\cancel{\partial\mathcal{S}}}$ defined in (1.23) according to the following formula:

$$\forall h,x\in\Omega,\quad \forall\vartheta\in\mathbb{R},\quad D_x\psi^0_{\cancel{\partial\mathcal{S}}}(h,x)\cdot\zeta_\vartheta = \psi^1_{\cancel{\partial\mathcal{S}}}(\vartheta,x,h). \tag{5.19}$$

Proof of (5.19). The relation (5.19) can be proved as follows. We first recall that $\psi^0_{\cancel{\partial\mathcal{S}}}$ is symmetric in its variables. Indeed by uniqueness of the Dirichlet problem (1.23) we have for any $h\in\Omega$, the decomposition in Ω:

$$\psi^0_{\cancel{\partial\mathcal{S}}}(h,\cdot) = G(\cdot - h) + G_\Omega(h,\cdot), \tag{5.20}$$

where G_Ω denotes the Green function associated with the domain Ω and the homogeneous Dirichlet condition, that is

$$\Delta G_\Omega(h,\cdot) = \delta_h \text{ in } \Omega,\quad G_\Omega(h,\cdot) = 0 \text{ on } \partial\Omega.$$

Using the decomposition (5.20), that the Newtonian potential G is even and the symmetry of G_Ω we get that the function $\psi^0_{\cancel{\partial\mathcal{S}}}$ is symmetric with respect to its arguments, that is:

$$\forall h,x\in\Omega,\quad \psi^0_{\cancel{\partial\mathcal{S}}}(h,x) = \psi^0_{\cancel{\partial\mathcal{S}}}(x,h). \tag{5.21}$$

It follows that $(D_x\psi^0_{\cancel{\partial\mathcal{S}}})(x,h)\cdot\zeta_\vartheta = (D_h\psi^0_{\cancel{\partial\mathcal{S}}})(h,x)\cdot\zeta_\vartheta$. Next we observe that $D_h\psi^0_{\cancel{\partial\mathcal{S}}}(h,\cdot)\cdot\zeta_\vartheta$ satisfies the same Dirichlet problem as $\psi^1_{\cancel{\partial\mathcal{S}}}(\vartheta,h,\cdot)$, by derivation of (1.23). Formula (5.19) follows then from the uniqueness of solutions to the Dirichlet problem, after switching h and x. $\square$

- *Definition of* $\psi^1_{\cancel{\partial\Omega}}(q,\cdot)$ *and of* $P^1(q,X)$. Let us denote, for any $q := (\vartheta,h)$ in $\mathbb{R}\times\Omega$, by $P^1(q,X)$ the polynomial defined by

$$P^1(q,X) := -\frac{1}{2}\langle R(\vartheta)^t\, D^2_x\psi^0_{\cancel{\partial\mathcal{S}}}(h,h)R(\vartheta), T^2(\mathsf{p}^{-1}_{\cancel{\partial\Omega}}) + X^{\otimes 2}\rangle_{\mathbb{R}^{2\times 2}} + R(\vartheta)^t\, D_x\psi^1_{\cancel{\partial\mathcal{S}}}(q,h)\cdot\big(\zeta - X\big). \tag{5.22}$$

Above $D^2_x\psi^0_{\cancel{\partial\mathcal{S}}}(h,h)$ denotes the second derivative of $\psi^0_{\cancel{\partial\mathcal{S}}}(h,\cdot)$ evaluated in h, $D_x\psi^1_{\cancel{\partial\mathcal{S}}}(q,h)$ stands for the derivative of $\psi^1_{\cancel{\partial\mathcal{S}}}(q,\cdot)$ evaluated in h and $X^{\otimes 2}$ stands for the 2×2 matrix $X\otimes X$. The notation $T^2(\mathsf{p}^{-1}_{\cancel{\partial\Omega}})$ stands for

$$T^2(\mathsf{p}^{-1}_{\cancel{\partial\Omega}}) := \int_{\partial\mathcal{S}_0}\frac{\partial\psi^{-1}_{\cancel{\partial\Omega}}}{\partial n}(X)\,X^{\otimes 2}\,\mathrm{d}s(X). \tag{5.23}$$

This notation is justified by (5.6).

Observe that $P^1(q,X)$ is harmonic (since every monomial of Taylor's expansions of harmonic functions are themselves harmonic) and satisfies

$$\int_{\partial\mathcal{S}_0} P^1(q,\cdot)\mathsf{p}^{-1}_{\cancel{\partial\Omega}}\,\mathrm{d}s = 0. \tag{5.24}$$

Therefore, according to Corollary 7 there exists a unique smooth function $\psi^1_{\cancel{\partial\Omega}}(q,\cdot)$ satisfying

$$-\Delta\psi^1_{\cancel{\partial\Omega}}(q,\cdot) = 0 \quad \text{in } \mathbb{R}^2\setminus\mathcal{S}_0, \tag{5.25a}$$

$$\psi^1_{\cancel{\partial\Omega}}(q,\cdot) = P^1(q,\cdot) \quad \text{on } \partial\mathcal{S}_0, \tag{5.25b}$$

and vanishing at infinity. Moreover

$$\int_{\partial\mathcal{S}_0} \frac{\partial\psi^1_{\cancel{\partial\Omega}}}{\partial n}(q,\cdot)\,\mathrm{d}s = 0. \tag{5.25c}$$

Our main result regarding the potential ψ^ε is the following, in addition to Lemma 6.

Proposition 7. *Let $\delta > 0$. There exists $\varepsilon_0 \in (0,1)$ and $\mathrm{p}_{\partial\mathcal{S}_0,r} \in L^\infty\big(\mathfrak{Q}_{\delta,\varepsilon_0}; L^2(\partial\mathcal{S}_0;\mathbb{R})\big)$, depending only on $\mathcal{S}_0$ and Ω, such that for any $(\varepsilon, q) \in \mathfrak{Q}_{\delta,\varepsilon_0}$ and for any $X \in \partial\mathcal{S}_0$,*

$$\begin{aligned}\frac{\partial\psi^\varepsilon}{\partial n}(q,\varepsilon R(\vartheta)X + h) = \frac{1}{\varepsilon}\frac{\partial\psi^{-1}_{\cancel{\partial\Omega}}}{\partial n}(X) + \left(\frac{\partial\psi^0_{\cancel{\partial\Omega}}}{\partial n}(q,X) - R(\vartheta)^t\, u_\Omega(h)\cdot\tau\right)&\\ + \varepsilon\left(\frac{\partial\psi^1_{\cancel{\partial\Omega}}}{\partial n} - \frac{\partial P^1}{\partial n}\right)(q,X) + \varepsilon^2\mathrm{p}_{\partial\mathcal{S}_0,r}(\varepsilon,q,X).&\end{aligned} \tag{5.26}$$

We recall that the set $\mathfrak{Q}_{\delta,\varepsilon_0}$ was defined in (2.4).

The proofs of Lemma 6 and of Proposition 7 are gathered in Section 5.3.

5.2.2. *Potential part.* For any $j = 1,2,3$, for any q in $\mathcal{Q}$, we consider the functions $K^\varepsilon_j(q,\cdot)$ on $\partial\Omega \cup \partial\mathcal{S}^\varepsilon(q)$ given by:

$$K^\varepsilon_j(q,\cdot) := n\cdot\xi_j(q,\cdot) \text{ on } \partial\Omega\cup\partial\mathcal{S}^\varepsilon(q), \tag{5.27}$$

where n denotes the unit normal to $\partial\mathcal{S}^\varepsilon(q)\cup\partial\Omega$, pointing outside $\mathcal{F}^\varepsilon(q)$ and the functions $\xi_j(q,\cdot)$ are given by the formula (1.5).

Then the Kirchhoff's potentials $\varphi^\varepsilon_j(q,\cdot)$, for $j = 1,2,3$, are the unique (up to an additive constant) solutions in $\mathcal{F}^\varepsilon(q)$ of the following Neumann problem:

$$\Delta\varphi^\varepsilon_j(q,\cdot) = 0 \qquad \text{in } \mathcal{F}^\varepsilon(q), \tag{5.28a}$$

$$\frac{\partial\varphi^\varepsilon_j}{\partial n}(q,\cdot) = K^\varepsilon_j(q,\cdot) \qquad \text{on } \partial\mathcal{S}^\varepsilon(q), \tag{5.28b}$$

$$\frac{\partial\varphi^\varepsilon_j}{\partial n}(q,\cdot) = 0 \qquad \text{on } \partial\Omega. \tag{5.28c}$$

The functions $K^\varepsilon_j(q,\cdot)$ (respectively $\varphi^\varepsilon_j(q,\cdot)$) are the counterpart, for the case where the size of the solid is of order ε, of the functions defined in (1.6) (resp. in (1.7)) in the case where the size of the solid is of order 1.

We will use the vector notations:

$$\boldsymbol{\varphi}^\varepsilon = (\varphi^\varepsilon_1,\, \varphi^\varepsilon_2,\, \varphi^\varepsilon_3)^t \quad\text{and}\quad \boldsymbol{K}^\varepsilon = (K^\varepsilon_1,\, K^\varepsilon_2,\, K^\varepsilon_3)^t. \tag{5.29}$$

Our result on the expansion of the Kirchoff potentials φ^ε_j is the following.

Proposition 8. *Let $\delta > 0$. There exists $\varepsilon_0 \in (0,1)$ and*

(i) there exists $\boldsymbol{\varphi}_r \in L^\infty\big(\mathfrak{Q}_{\delta,\varepsilon_0}; L^2(\partial\mathcal{S}_0;\mathbb{R}^3)\big)$ and $\check{\mathbf{c}} \in L^\infty\big(\mathfrak{Q}_{\delta,\varepsilon_0};\mathbb{R}^3\big)$ such that for any $(\varepsilon,q)\in\mathfrak{Q}_{\delta,\varepsilon_0}$, with $q = (\vartheta, h)$, for any $X \in \partial\mathcal{S}_0$,

$$\boldsymbol{\varphi}^\varepsilon(q,\varepsilon R(\vartheta)X + h) = \varepsilon I_\varepsilon \mathcal{R}(\vartheta)\Big(\boldsymbol{\varphi}_{\cancel{\partial\Omega}}(X) + \check{\mathbf{c}}(\varepsilon,q) + \varepsilon^2\boldsymbol{\varphi}_r(\varepsilon,q,X)\Big), \tag{5.30}$$

(ii) there exists $\mathrm{p}_{\partial\mathcal{S}_0,r} \in L^\infty\big(\mathfrak{Q}_{\delta,\varepsilon_0}; L^2(\partial\mathcal{S}_0;\mathbb{R}^3)\big)$ such that for any $(\varepsilon,q)\in\mathfrak{Q}_{\delta,\varepsilon_0}$, with $q = (\vartheta,h)$, for any $X\in\partial\mathcal{S}_0$,

$$\mathcal{R}(\vartheta)^t\,\frac{\partial\boldsymbol{\varphi}^\varepsilon}{\partial\tau}(q,\varepsilon R(\vartheta)X + h) = I_\varepsilon\Big(\frac{\partial\boldsymbol{\varphi}_{\cancel{\partial\Omega}}}{\partial\tau}(X) + \varepsilon^2\mathrm{p}_{\partial\mathcal{S}_0,r}(\varepsilon,q,X)\Big), \tag{5.31}$$

(iii) there exists $\mathrm{p}_{\partial\Omega,r} \in L^\infty\big(\mathfrak{Q}_{\delta,\varepsilon_0}; L^2(\partial\Omega;\mathbb{R}^3)\big)$ *such that for any* $(\varepsilon,q) \in \mathfrak{Q}_{\delta,\varepsilon_0}$*, for any* $x \in \partial\Omega$*,*

$$\frac{\partial \boldsymbol{\varphi}^\varepsilon}{\partial \tau}(q,x) = I_\varepsilon\, \varepsilon^2\, \mathrm{p}_{\partial\Omega,r}(\varepsilon,q,x). \tag{5.32}$$

Moreover the reminders $\boldsymbol{\varphi}_r$*,* $\mathrm{p}_{\partial\mathcal{S}_0,r}$ *and* $\mathrm{p}_{\partial\Omega,r}$ *depend only on* $\mathcal{S}_0$ *and* Ω*.*

The proof of Proposition 8 is given in Section 5.5.

5.3. **Proof of Proposition 7 and of Lemma 6.** We now turn to the proof of Proposition 7 and of Lemma 6. We proceed in four steps that we now detail. We rely on intermediate results: Lemma 13, Lemma 14, Lemma 16, Lemma 17, and Lemma 18 whose proofs are postponed to Subsection 5.4.

We will use the following functional space: for $-\frac{1}{2} \leqslant s \leqslant 1$, let the Hilbert space

$$F_s := H^s(\partial\mathcal{S}_0) \times H^s(\partial\Omega) \times \mathbb{R}.$$

We will mainly make use of the indices $s = 0$ and 1 and also for technical reasons of $-\frac{1}{2}$ and $\frac{1}{2}$.

First Step. Reduction to integral equations. We look for the solution $\psi^\varepsilon(q,\cdot)$ of (5.14) as a superposition of two single-layer integrals, one supported on the body's boundary and the other one supported on $\partial\Omega$. This transforms (5.14) in an integral system as follows.

We define, for any $(\varepsilon,q) \in \mathfrak{Q}$ with $q = (\vartheta,h) \in \mathbb{R}\times\Omega$, two operators $K_{\partial\mathcal{S}_0}(\varepsilon,q)$ and $K_{\partial\Omega}(\varepsilon,q)$ respectively from $L^2(\partial\Omega)$ to $H^1(\partial\mathcal{S}_0)$ and from $L^2(\partial\mathcal{S}_0)$ to $H^1(\partial\Omega)$, by the following formulas: given densities $\mathrm{p}_{\partial\Omega}$ and $\mathrm{p}_{\partial\mathcal{S}}$ respectively in $L^2(\partial\Omega)$ and $L^2(\partial\mathcal{S}_0)$,

$$K_{\partial\mathcal{S}_0}(\varepsilon,q)[\mathrm{p}_{\partial\Omega}](\cdot) := SL[p_{\partial\Omega}](\varepsilon R(\vartheta)\cdot + h) \qquad \text{on } \partial\mathcal{S}_0, \tag{5.33}$$

$$K_{\partial\Omega}(\varepsilon,q)[\mathrm{p}_{\partial\mathcal{S}_0}](\cdot) := \int_{\partial\mathcal{S}_0} \mathrm{p}_{\partial\mathcal{S}_0}(Y) G\big(\cdot - (\varepsilon R(\vartheta)Y + h)\big)\mathrm{d}s(Y) \qquad \text{on } \partial\Omega. \tag{5.34}$$

Thanks to Lemma 12 (ii), the operators $K_{\partial\mathcal{S}_0}(\varepsilon,q)$ and $K_{\partial\Omega}(\varepsilon,q)$ are compact respectively from $L^2(\partial\Omega)$ to $H^1(\partial\mathcal{S}_0)$ and from $L^2(\partial\mathcal{S}_0)$ to $H^1(\partial\Omega)$.

We also introduce for $(\varepsilon,q) \in \mathfrak{Q}$, the operator $\mathfrak{A}(\varepsilon,q) : F_0 \to F_1$ as follows: for any $\mathfrak{p} := (\mathrm{p}_{\partial\mathcal{S}_0}, \mathrm{p}_{\partial\Omega}, C)$ in F_0,

$$\mathfrak{A}(\varepsilon,q)[\mathfrak{p}] := \Big(SL[\mathrm{p}_{\partial\mathcal{S}_0}] + K_{\partial\mathcal{S}_0}(\varepsilon,q)[\mathrm{p}_{\partial\Omega}] - C, SL[\mathrm{p}_{\partial\Omega}] + K_{\partial\Omega}(\varepsilon,q)[\mathrm{p}_{\partial\mathcal{S}_0}], \int_{\partial\mathcal{S}_0} \mathrm{p}_{\partial\mathcal{S}_0}\mathrm{d}s\Big). \tag{5.35}$$

Let us observe that in order to simplify the notations, we omitted to write the trace operators applied to the single-layers in $K_{\partial\mathcal{S}_0}(\varepsilon,q)$, $K_{\partial\Omega}(\varepsilon,q)$ and $\mathfrak{A}(\varepsilon,q)$. We also emphasize that the dependence of $\mathfrak{A}(\varepsilon,q)$ on (ε,q) occurs only through the compact operators $K_{\partial\mathcal{S}_0}(\varepsilon,q)$ and $K_{\partial\Omega}(\varepsilon,q)$.

Now the equation (5.14) is transformed into an integral system thanks to the following lemma.

Lemma 13. *For any* $(\varepsilon,q) \in \mathfrak{Q}$*, let* $\mathfrak{p}^\varepsilon(q,\cdot) = \big(p^\varepsilon_{\partial\mathcal{S}_0}(q,\cdot), p^\varepsilon_{\partial\Omega}(q,\cdot), C^\varepsilon(q)\big) \in F_0$ *such that*

$$\mathfrak{A}(\varepsilon,q)\big[\mathfrak{p}^\varepsilon(q,\cdot) + \big(0,0,G(\varepsilon)\big)\big] = (0,0,-1). \tag{5.36}$$

Then the function in $\mathcal{F}^\varepsilon(q)$

$$\psi^\varepsilon(q,\cdot) := SL[p^\varepsilon_{\partial\mathcal{S}^\varepsilon(q)}(q,\cdot)] + SL[p^\varepsilon_{\partial\Omega}(q,\cdot)], \tag{5.37}$$

where the density $p^\varepsilon_{\partial\mathcal{S}^\varepsilon(q)}(q,\cdot)$ *on* $\partial\mathcal{S}^\varepsilon(q)$ *is defined through the relation:*

$$\textit{for } X \in \partial\mathcal{S}_0, \quad p^\varepsilon_{\partial\mathcal{S}_0}(q,X) := \varepsilon p^\varepsilon_{\partial\mathcal{S}^\varepsilon(q)}(q,\varepsilon R(\vartheta)X + h), \tag{5.38}$$

is the solution of (5.14). *Moreover the normal derivative* $\frac{\partial \psi^\varepsilon}{\partial n}(q,\cdot)$ *on* $\partial\mathcal{S}^\varepsilon(q)$ *is given by:*

$$\text{for } X \in \partial\mathcal{S}_0, \quad \frac{\partial \psi^\varepsilon}{\partial n}(q, \varepsilon R(\vartheta)X + h) = \frac{1}{\varepsilon} p^\varepsilon_{\partial\mathcal{S}_0}(q, X). \tag{5.39}$$

The proof of Lemma 13 is postponed to Section 5.4.1.

Second Step. Construction of an approximate solution. In this step we describe an approximation $\mathfrak{p}_{\text{app}}$ up to order $\mathcal{O}(\varepsilon^3)$ of the solution $\mathfrak{p}^\varepsilon$ of (5.36) and reformulate the equation (5.36) in terms of the rest $\mathfrak{p}^\varepsilon - \mathfrak{p}_{\text{app}}$.

We introduce the various terms involved in the approximation.

- *Densities on* $\partial\mathcal{S}_0$. We first introduce the following potentials defined in $\mathbb{R}^2 \setminus \partial\mathcal{S}_0$:

  - $\psi^{-1}_{\cancel{\partial\Omega}}$ defined in (2.16), extended by $C_{\cancel{\partial\Omega}}$ in $\mathcal{S}_0$,
  - $\psi^0_{\cancel{\partial\Omega}}(h,\cdot)$ defined in (5.17), extended by $P^0(h,\cdot)$ in $\mathcal{S}_0$ and
  - $\psi^1_{\cancel{\partial\Omega}}(q,\cdot)$ defined in (5.25), extended by $P^1(q,\cdot)$ in $\mathcal{S}_0$.

  Now we let $\mathsf{p}^0_{\cancel{\partial\Omega}}(q,\cdot)$ and $\mathsf{p}^1_{\cancel{\partial\Omega}}(q,\cdot)$ be the densities on $\partial\mathcal{S}_0$ respectively associated respectively with $\psi^0_{\cancel{\partial\Omega}}(h,\cdot)$ and $\psi^1_{\cancel{\partial\Omega}}(q,\cdot)$ Correspondingly, we write:

$$\psi^0_{\cancel{\partial\Omega}}(q,\cdot) = SL[\mathsf{p}^0_{\cancel{\partial\Omega}}(q,\cdot)], \tag{5.40a}$$

$$\psi^1_{\cancel{\partial\Omega}}(q,\cdot) = SL[\mathsf{p}^1_{\cancel{\partial\Omega}}(q,\cdot)], \tag{5.40b}$$

  and we observe that the first identity of (5.6) is also true in $\mathcal{S}_0$.
- *Densities on* $\partial\Omega$. We introduce the following potentials defined in $\mathbb{R}^2 \setminus \partial\Omega$:

  - $\psi^0_{\cancel{\partial\mathcal{S}}}(h,\cdot)$, defined in (1.23), extended by $G(\cdot - h)$ in $\mathbb{R}^2 \setminus \Omega$,
  - $\psi^1_{\cancel{\partial\mathcal{S}}}(q,\cdot)$, defined in (5.18), extended by $-(\nabla G)(\cdot - h)\cdot\zeta_\vartheta$ in $\mathbb{R}^2 \setminus \Omega$, and
  - $\psi^2_{\cancel{\partial\mathcal{S}}}(q,\cdot)$, defined below in (5.44); extended by $Q^2(q,\cdot)$, defined in (5.42), in $\mathbb{R}^2 \setminus \Omega$.

  We let $\mathsf{p}^0_{\cancel{\partial\mathcal{S}}}(q,\cdot)$, $\mathsf{p}^1_{\cancel{\partial\mathcal{S}}}(q,\cdot)$ and $\mathsf{p}^2_{\cancel{\partial\mathcal{S}}}(q,\cdot)$ be the densities on $\partial\Omega$ respectively associated with $\psi^0_{\cancel{\partial\mathcal{S}}}(h,\cdot)$, $\psi^1_{\cancel{\partial\mathcal{S}}}(q,\cdot)$ and $\psi^2_{\cancel{\partial\mathcal{S}}}(q,\cdot)$. This translates into:

$$\psi^0_{\cancel{\partial\mathcal{S}}}(h,\cdot) = SL[\mathsf{p}^0_{\cancel{\partial\mathcal{S}}}(h,\cdot)], \tag{5.41a}$$

$$\psi^1_{\cancel{\partial\mathcal{S}}}(q,\cdot) = SL[\mathsf{p}^1_{\cancel{\partial\mathcal{S}}}(q,\cdot)], \tag{5.41b}$$

$$\psi^2_{\cancel{\partial\mathcal{S}}}(q,\cdot) = SL[\mathsf{p}^2_{\cancel{\partial\mathcal{S}}}(q,\cdot)]. \tag{5.41c}$$

- *Definition of* $\psi^2_{\cancel{\partial\mathcal{S}}}(q,\cdot)$ *and of* $Q^2(q,\cdot)$. In order to define $\psi^2_{\cancel{\partial\mathcal{S}}}(q,\cdot)$, we introduce the harmonic function in $\mathbb{R}^2 \setminus \Omega$:

$$Q^2(q,x) := \frac{1}{2}\langle R(\vartheta)^t\, D^2G(x-h)R(\vartheta), T^2(\mathsf{p}^{-1}_{\cancel{\partial\Omega}})\rangle_{\mathbb{R}^{2\times 2}} + R(\vartheta)^t\,\nabla G(x-h)\cdot T^1(\mathsf{p}^0_{\cancel{\partial\Omega}}(q,\cdot)), \tag{5.42}$$

  where

$$T^1(\mathsf{p}^0_{\cancel{\partial\Omega}}(q,\cdot)) := \int_{\partial\mathcal{S}_0} Y\,\mathsf{p}^0_{\cancel{\partial\Omega}}(q,Y)\,\mathrm{d}s(Y). \tag{5.43}$$

Then we consider $\psi^2_{\cancel{\partial\mathcal{S}}}(q,\cdot)$ as the solution of

$$-\Delta\psi^2_{\cancel{\partial\mathcal{S}}}(q,\cdot) = 0 \quad \text{in } \Omega, \tag{5.44a}$$

$$\psi^2_{\cancel{\partial\mathcal{S}}}(q,\cdot) = Q^2(q,\cdot) \quad \text{on } \partial\Omega, \tag{5.44b}$$

extended by $Q^2(q,x)$ for x in $\mathbb{R}^2\setminus\Omega$. These functions $\psi^2_{\cancel{\partial\mathcal{S}}}(q,\cdot)$ and $Q^2(q,\cdot)$ do not appear in the claim of Proposition 7 and of Lemma 6 but will be useful later.

With the choices above we expect to construct a solution of (5.36) with $p^\varepsilon_{\partial\mathcal{S}_0}(\varepsilon,q,\cdot)$ and $p^\varepsilon_{\partial\Omega}(\varepsilon,q,\cdot)$ respectively close to

$$\begin{aligned} \mathsf{p}_{\partial\mathcal{S}_0,\mathrm{app}}(\varepsilon,q,\cdot) &:= \mathsf{p}^{-1}_{\cancel{\partial\Omega}} + \varepsilon\mathsf{p}^0_{\cancel{\partial\Omega}}(q,\cdot) + \varepsilon^2\,\mathsf{p}^1_{\cancel{\partial\Omega}}(q,\cdot), \\ \mathsf{p}_{\partial\Omega,\mathrm{app}}(\varepsilon,q,\cdot) &:= \mathsf{p}^0_{\cancel{\partial\mathcal{S}}}(h,\cdot) + \varepsilon\mathsf{p}^1_{\cancel{\partial\mathcal{S}}}(q,\cdot) + \varepsilon^2\mathsf{p}^2_{\cancel{\partial\mathcal{S}}}(q,\cdot). \end{aligned} \tag{5.45}$$

The corresponding approximation $C_{\mathrm{app}}(\varepsilon,q)$ of $C^\varepsilon(q)$ is chosen as:

$$C_{\mathrm{app}}(\varepsilon,q) := -G(\varepsilon) + \mathsf{C}^0(h) + \varepsilon\mathsf{C}^1(q) + \varepsilon^2\mathsf{C}^2(q), \tag{5.46}$$

with

$$\begin{aligned} \mathsf{C}^0(h) &:= C_{\cancel{\partial\Omega}} + \psi^0_{\cancel{\partial\mathcal{S}}}(h,h), \\ \mathsf{C}^1(q) &:= 2D_x\psi^0_{\cancel{\partial\mathcal{S}}}(h,h)\cdot\zeta_\vartheta, \\ \mathsf{C}^2(q) &:= \psi^2_{\cancel{\partial\mathcal{S}}}(q,h) + D_x\psi^1_{\cancel{\partial\mathcal{S}}}(q,h)\cdot\zeta_\vartheta - \frac{1}{2}\langle R(\vartheta)^t\,D_x^2\psi^0_{\cancel{\partial\mathcal{S}}}(h,h)R(\vartheta), T^2(\mathsf{p}^{-1}_{\cancel{\partial\Omega}})\rangle_{\mathbb{R}^{2\times 2}}. \end{aligned} \tag{5.47}$$

Using that $\psi^0_{\cancel{\partial\mathcal{S}}}$ is symmetric with respect to its two arguments (see (5.21)), and using (1.25), we see that the first terms of the expansion above are the same as those claimed in (3.17), that is

$$\mathsf{C}^0(h) := C_{\cancel{\partial\Omega}} + 2\psi_\Omega(h) \quad \text{and} \quad \mathsf{C}^1(q) := 2\psi_c(q). \tag{5.48}$$

We finally denote

$$\mathfrak{p}_{\mathrm{app}}(\varepsilon,q,\cdot) := \big(\mathsf{p}_{\partial\mathcal{S}_0,\mathrm{app}}(\varepsilon,q,\cdot), \mathsf{p}_{\partial\Omega,\mathrm{app}}(\varepsilon,q,\cdot), C_{\mathrm{app}}(\varepsilon,q)\big).$$

Now the equation (5.36) translates as follows. Let us introduce $g_{\partial\mathcal{S}_0}(\varepsilon,q,\cdot)$ and $g_{\partial\Omega}(\varepsilon,q,\cdot)$ two functions respectively defined on $\partial\mathcal{S}_0$ and $\partial\Omega$, for $q=(\vartheta,h)$, by

(5.49a)

$$-g_{\partial\mathcal{S}_0}(\varepsilon,q,\cdot) := \sum_{j=0}^{2}\int_{\partial\Omega}\mathsf{p}^j_{\cancel{\partial\mathcal{S}}}(q,y)\eta_{3-j}(\varepsilon,q,\cdot,y)\mathrm{d}s(y),$$

(5.49b)

$$-g_{\partial\Omega}(\varepsilon,q,x) := \int_{\partial\mathcal{S}_0}\mathsf{p}^{-1}_{\cancel{\partial\Omega}}(y)\eta_3(\varepsilon,(\vartheta,x),-y,h)\mathrm{d}s(y) + \sum_{j=0}^{1}\int_{\partial\mathcal{S}_0}\mathsf{p}^j_{\cancel{\partial\Omega}}(q,y)\eta_{2-j}(\varepsilon,(\vartheta,x),-y,h)\mathrm{d}s(y),$$

where we have denoted, for $N\geqslant 1$,

$$\eta_N(\varepsilon,q,\cdot,y) := \int_0^1\frac{(1-\sigma)^{N-1}}{(N-1)!}D^NG(\sigma\varepsilon R(\vartheta)\cdot+h-y)\cdot(R(\vartheta)\cdot)^{\otimes N}\mathrm{d}\sigma. \tag{5.50}$$

Let

$$\mathfrak{g}(\varepsilon,q,\cdot) := \big(g_{\partial\mathcal{S}_0}(\varepsilon,q,\cdot), g_{\partial\Omega}(\varepsilon,q,\cdot), 0\big). \tag{5.51}$$

We can deduce from the definitions of the densities $\mathsf{p}^j_{\partial\mathcal{S}}$ for $j=0,1,2$, $\mathsf{p}^{-1}_{\partial\Omega}$ and $\mathsf{p}^j_{\partial\Omega}$ for $j=0$ and 1, and from Lemma 12, (ii) that $g_{\partial\mathcal{S}_0}(\varepsilon,q,\cdot)$ and $g_{\partial\Omega}(\varepsilon,q,\cdot)$ belong respectively to $H^1(\partial\mathcal{S}_0)$ and to $H^1(\partial\Omega)$. Actually we even have

$$\mathfrak{g}\in L^\infty(\mathfrak{Q}_\delta;F_1). \tag{5.52}$$

We can now state the result of this step.

Lemma 14. *For any $(\varepsilon,q)\in\mathfrak{Q}$, let $\mathfrak{p}_r(\varepsilon,q,\cdot)\in F_0$ satisfying:*

$$\mathfrak{A}(\varepsilon,q)[\mathfrak{p}_r(\varepsilon,q,\cdot)]=\mathfrak{g}(\varepsilon,q,\cdot). \tag{5.53}$$

Then

$$\mathfrak{p}^\varepsilon(q,\cdot):=\mathfrak{p}_{app}(\varepsilon,q,\cdot)+\varepsilon^3\,\mathfrak{p}_r(\varepsilon,q,\cdot), \tag{5.54}$$

is solution of (5.36).

The proof of Lemma 14 is postponed to Section 5.4.2.

Let us stress in particular that the third argument of the left hand side of (5.53) does not contain the singular term $G(\varepsilon)$ anymore and that the third argument of the right hand side of (5.53) is now 0.

Third Step. Existence and estimate of the reminders. In this third step we prove, for $(\varepsilon,q)\in\mathfrak{Q}_{\delta,\varepsilon_0}$, with δ and ε_0 positive small enough, the existence of $\mathfrak{p}_r(\varepsilon,q,\cdot)\in F_0$ satisfying (5.53) and provide an estimate in F_0, uniform over $(\varepsilon,q)\in\mathfrak{Q}_{\delta,\varepsilon_0}$.

We will make use of the fact that the the third argument of the right hand side of (5.53) vanishes. Accordingly, we denote

$$\tilde{F}_1:=H^1(\partial\mathcal{S}_0)\times H^1(\partial\Omega)\times\{0\}, \tag{5.55}$$

which is a closed subspace of F_1, and prove the following result.

Lemma 15. *Let $\delta>0$. There exists ε_0 in $(0,1)$, such that for any $\mathfrak{g}$ in $L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0};\tilde{F}_1)$, there exists $\mathfrak{p}_r$ in $L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0};F_0)$ such that $\mathfrak{p}_r(\varepsilon,q,\cdot)$ solves* (5.53) *for any $(\varepsilon,q)\in\mathfrak{Q}_{\delta,\varepsilon_0}$.*

Proof of Lemma 15. In order to prove Lemma 15 let us start with stating a perturbative result. The framework is as follows. Given X and Y two Banach spaces, we denote $\mathcal{L}(X;Y)$ the space of bounded linear operators from X to Y. Let $\delta>0$. We introduce the following families of operators.

- First we consider a family of operators in $\mathcal{L}(L^2(\partial\Omega);H^1(\partial\mathcal{S}_0))$:

$$\tilde{K}_{\partial\mathcal{S}_0}\in\mathrm{Lip}\left(\overline{\Omega_\delta};\mathcal{L}\big(L^2(\partial\Omega);H^1(\partial\mathcal{S}_0)\big)\right)\text{ such that for all }h\text{ in }\overline{\Omega_\delta},\quad \tilde{K}_{\partial\mathcal{S}_0}(h)\text{ is compact from }L^2(\partial\Omega)\text{ to }H^1(\partial\mathcal{S}_0). \tag{5.56}$$

- Next we consider two families of operators: one in $\mathcal{L}(L^2(\partial\Omega);H^1(\partial\mathcal{S}_0))$ and the other one in $\mathcal{L}(L^2(\partial\mathcal{S}_0);H^1(\partial\Omega))$:

$$(T_{\partial\mathcal{S}_0}(\varepsilon,q))_{(\varepsilon,q)\in\mathfrak{Q}_\delta}\text{ bounded in }\mathcal{L}(L^2(\partial\Omega);H^1(\partial\mathcal{S}_0)), \tag{5.57a}$$

$$(T_{\partial\Omega}(\varepsilon,q))_{(\varepsilon,q)\in\mathfrak{Q}_\delta}\text{ bounded in }\mathcal{L}(L^2(\partial\mathcal{S}_0);H^1(\partial\Omega)). \tag{5.57b}$$

Given these operators we can construct the following one. For $(\varepsilon,q)\in\mathfrak{Q}_\delta$, let $A(\varepsilon,q):F_0\to F_1$ given by the following formula: for any $\mathfrak{p}:=(\mathfrak{p}_{\partial\mathcal{S}_0},\mathfrak{p}_{\partial\Omega},C)\in F_0$,

$$A(\varepsilon,q)[\mathfrak{p}]:=\big(A(\varepsilon,q)[\mathfrak{p}]_i\big)_{1\leqslant i\leqslant3}\in F_1, \tag{5.58}$$

with

$$A(\varepsilon,q)[\mathfrak{p}]_1 := SL[\mathrm{p}_{\partial\mathcal{S}_0}] + \tilde{K}_{\partial\mathcal{S}_0}(h)[\mathrm{p}_{\partial\Omega}] + \varepsilon T_{\partial\mathcal{S}_0}(\varepsilon,q)[\mathrm{p}_{\partial\Omega}] - C, \tag{5.59a}$$

$$A(\varepsilon,q)[\mathfrak{p}]_2 := SL[\mathrm{p}_{\partial\Omega}] + \varepsilon T_{\partial\Omega}(\varepsilon,q)[\mathrm{p}_{\partial\mathcal{S}_0}], \tag{5.59b}$$

$$A(\varepsilon,q)[\mathfrak{p}]_3 := \int_{\partial\mathcal{S}_0} \mathrm{p}_{\partial\mathcal{S}_0}\mathrm{d}s. \tag{5.59c}$$

Our perturbative result is as follows.

Lemma 16. *Let $\delta>0$ and for $(\varepsilon,q)\in\mathfrak{Q}_\delta$, $A(\varepsilon,q)$ given as above, with assumptions* (5.56) *and* (5.57)*. Then there exists $\varepsilon_0\in(0,1)$ such that for any $(\varepsilon,q)\in\mathfrak{Q}_{\delta,\varepsilon_0}$, $A(\varepsilon,q)$ is an isomorphism from F_0 to F_1 and*

$$\sup_{(\varepsilon,q)\in\mathfrak{Q}_{\delta,\varepsilon_0}} \|A(\varepsilon,q)^{-1}\|_{\mathcal{L}(F_1;F_0)} < \infty. \tag{5.60}$$

The proof of Lemma 16 is postponed to Section 5.4.3.

In our case, Lemma 16 is applied as follows. We define, for any $(\varepsilon,q)\in\mathfrak{Q}$, with $q=(\vartheta,h)$,

- for any density $\mathrm{p}_{\partial\Omega}\in L^2(\partial\Omega)$,
$$\tilde{K}_{\partial\mathcal{S}_0}(h)[\mathrm{p}_{\partial\Omega}] := K_{\partial\mathcal{S}_0}(0,0,h)[\mathrm{p}_{\partial\Omega}] = SL[\mathrm{p}_{\partial\Omega}](h) \ \text{ as a constant function on } \partial\mathcal{S}_0,$$
$$T_{\partial\mathcal{S}_0}(\varepsilon,q)[\mathrm{p}_{\partial\Omega}] := \int_{\partial\Omega} \mathrm{p}_{\partial\Omega}(y)\eta_1(\varepsilon,q,\cdot,y)\mathrm{d}s(y) \ \text{ on } \partial\mathcal{S}_0,$$
- for any density $\mathrm{p}_{\partial\mathcal{S}_0}\in L^2(\partial\mathcal{S}_0)$,
$$T_{\partial\Omega}(\varepsilon,q)[\mathrm{p}_{\partial\mathcal{S}_0}] := \int_{\partial\mathcal{S}_0} \mathrm{p}_{\partial\mathcal{S}_0}(Y)\eta_1(\varepsilon,\vartheta,\cdot,-Y,h)\mathrm{d}s(y) \ \text{ on } \partial\Omega.$$

The following lemma entails that the hypotheses of Lemma 16 are satisfied.

Lemma 17. *Let $\delta>0$. With the definitions above,* (5.56) *and* (5.57) *hold true.*

The proof of Lemma 17 is postponed to Section 5.4.4.

Then we consider the operator $A(\varepsilon,q)$ associated with these operators $\tilde{K}_{\partial\mathcal{S}_0}(h)$, $T_{\partial\mathcal{S}_0}(\varepsilon,q)$ and $T_{\partial\Omega}(\varepsilon,q)$ as given by (5.58)-(5.59). The next lemma shows that this operator $A(\varepsilon,q)$ provides the existence of a solution to (5.53) with uniform estimates.

Lemma 18. *Let $\delta>0$. There exists $\varepsilon_0\in(0,1)$ such that for any $(\varepsilon,q)\in\mathfrak{Q}_{\delta,\varepsilon_0}$,*

$$\mathfrak{p}_r(\varepsilon,q,\cdot) := A(\varepsilon,q)^{-1}\mathfrak{g}(\varepsilon,q,\cdot) \tag{5.61}$$

belongs to F_0 and solves (5.53)*. Moreover $\mathfrak{p}_r$ is in $L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0};F_0)$.*

The proof of Lemma 18 is postponed to Section 5.4.5. Now once assumed Lemma 18, Lemma 15 follows in a straightforward manner. □

Fourth Step. Conclusion.

End of proof of Lemma 6. We apply Lemma 15 to (5.51). Thanks to (5.52) the assumption is satisfied. Regarding $C^\varepsilon(q)$ this yields an expansion actually better than the one stated in Lemma 6, that is, according to (5.46) and (5.48) and what precedes, there exists $C_r\in L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0};\mathbb{R})$ such that

$$C^\varepsilon(q) = -G(\varepsilon) + C_{\cancel{\partial\Omega}} + \psi^0_{\cancel{\partial\mathcal{S}}}(h,h) + 2\varepsilon D_x\psi^0_{\cancel{\partial\mathcal{S}}}(h,h)\cdot\zeta_\vartheta + \varepsilon^2\,\mathsf{C}^2(q) + \varepsilon^3\,C_r(\varepsilon,q), \tag{5.62}$$

where $\mathsf{C}^2(q)$ is given by (5.47). In order to prove Lemma 6 it is therefore sufficient to observe that $\mathsf{C}^2(q)$ is bounded uniformly in $\mathbb{R}$ for $(\varepsilon, q) \in \mathfrak{Q}_\delta$ and to redefine $C_r(\varepsilon, q)$ such that $\varepsilon^2 C_r(\varepsilon, q)$ is equal to the sum of the two last terms in (5.62). ☐

End of proof of Proposition 7. Combining (5.39), (5.45) and (5.54), we get that on $\partial\mathcal{S}_0$

$$\frac{\partial\psi^\varepsilon}{\partial n}(q, \varepsilon R(\vartheta)\cdot + h) = \mathsf{p}^{-1}_{\partial\Omega}(\cdot) + \varepsilon \mathsf{p}^0_{\partial\Omega}(q, \cdot) + \varepsilon^2\, \mathsf{p}^1_{\partial\Omega}(q, \cdot) + \varepsilon^3\, \mathsf{p}_{\partial\mathcal{S}_0, r}(\varepsilon, q, \cdot),$$

with $\mathsf{p}_{\partial\mathcal{S}_0,r} \in L^\infty\big(\mathfrak{Q}_{\delta,\varepsilon_0}; L^2(\partial\mathcal{S}_0;\mathbb{R})\big)$. Moreover using (5.2) we have that

$$\mathsf{p}^{-1}_{\partial\Omega} = \frac{\partial\psi^{-1}_{\partial\Omega}}{\partial n}, \quad \mathsf{p}^0_{\partial\Omega} = \frac{\partial\psi^0_{\partial\Omega}}{\partial n} - \frac{\partial P^0}{\partial n} \text{ and } \mathsf{p}^1_{\partial\Omega} = \frac{\partial\psi^1_{\partial\Omega}}{\partial n} - \frac{\partial P^1}{\partial n}. \tag{5.63}$$

Referring to the definition of P^0 in (5.15) we obtain $\frac{\partial P^0}{\partial n}(q, X) = -R(\vartheta)^t\, u_\Omega(h)\cdot\tau$, for X on $\partial\mathcal{S}_0$, which concludes the proof of Proposition 7. ☐

5.4. **Proof of the intermediate lemmas.** In this subsection, we establish the intermediate lemmas used in Subsection 5.3.

5.4.1. *Proof of Lemma 13.* First observe that for any densities $p^\varepsilon_{\partial\mathcal{S}^\varepsilon(q)}(q,\cdot) \in H^{-\frac12}(\partial\mathcal{S}^\varepsilon(q))$ and $p^\varepsilon_{\partial\Omega}(q,\cdot) \in H^{-\frac12}(\partial\Omega)$, the right hand side of (5.37) is in $H^1_{loc}(\mathbb{R}^2)$ and harmonic in $\mathcal{F}^\varepsilon(q)$ and in $\mathbb{R}^2 \setminus \mathcal{F}^\varepsilon(q)$. In particular the equation (5.14a) is satisfied when $\psi^\varepsilon(q,\cdot)$ is given by (5.37) without further assumptions about $p^\varepsilon_{\partial\mathcal{S}^\varepsilon(q)}(q,\cdot)$ or $p^\varepsilon_{\partial\Omega}(q,\cdot)$.

Next we write (5.36) explicitly in the form:

$$\int_{\partial\mathcal{S}_0} p^\varepsilon_{\partial\mathcal{S}_0}(q,\cdot)\mathrm{d}s = -1, \tag{5.64a}$$

$$-G(\varepsilon) + SL[p^\varepsilon_{\partial\mathcal{S}_0}(q,\cdot)] + K_{\partial\mathcal{S}_0}(\varepsilon,q)[p^\varepsilon_{\partial\Omega}(q,\cdot)] = C^\varepsilon(q) \quad \text{on } \partial\mathcal{S}_0, \tag{5.64b}$$

$$SL[p^\varepsilon_{\partial\Omega}(q,\cdot)] + K_{\partial\Omega}(\varepsilon,q)[p^\varepsilon_{\partial\mathcal{S}_0}(q,\cdot)] = 0 \quad \text{on } \partial\Omega. \tag{5.64c}$$

Thanks to a change of variable, and using $G\big(\varepsilon(x-y)\big) = G(\varepsilon) + G(x-y)$, (5.33), (5.34) and (5.38), this can be recast as

$$\int_{\partial\mathcal{S}^\varepsilon(q)} p^\varepsilon_{\partial\mathcal{S}^\varepsilon(q)}(q,\cdot)\mathrm{d}s = -1, \tag{5.65a}$$

$$SL[p^\varepsilon_{\partial\mathcal{S}^\varepsilon(q)}(q,\cdot)] + SL[p^\varepsilon_{\partial\Omega}(q,\cdot)] = C^\varepsilon(q) \quad \text{on } \partial\mathcal{S}^\varepsilon(q), \tag{5.65b}$$

$$SL[p^\varepsilon_{\partial\mathcal{S}^\varepsilon(q)}(q,\cdot)] + SL[p^\varepsilon_{\partial\Omega}(q,\cdot)] = 0 \quad \text{on } \partial\Omega. \tag{5.65c}$$

In particular we infer from (5.65b) and (5.65c) that, when $\psi^\varepsilon(q,\cdot)$ is given by (5.37), with $\mathfrak{p}^\varepsilon(q,\cdot) = \big(p^\varepsilon_{\partial\mathcal{S}_0}(q,\cdot), p^\varepsilon_{\partial\Omega}(q,\cdot), C^\varepsilon(q)\big)$ solution of (5.36), the boundary conditions (5.14b) and (5.14c) are satisfied. Moreover, by uniqueness of the solutions to the Poisson problem:

$$\Delta\Psi = 0 \text{ in } \mathcal{S}^\varepsilon(q), \quad \Psi = C^\varepsilon(q) \text{ on } \partial\mathcal{S}^\varepsilon(q),$$

the right hand side of (5.37) is equal to $C^\varepsilon(q)$ in $\mathcal{S}^\varepsilon(q)$.

The single-layer potential $SL[p^\varepsilon_{\partial\Omega}(q,\cdot)]$ is smooth in a neighborhood of $\partial\mathcal{S}^\varepsilon(q)$. Hence, according to (5.2), when $\psi^\varepsilon(q,\cdot)$ is given by (5.37), the density $p^\varepsilon_{\partial\mathcal{S}^\varepsilon(q)}(q,\cdot)$ is equal to the jump across $\partial\mathcal{S}^\varepsilon(q)$ of the normal derivatives of the function equal to $\psi^\varepsilon(q,\cdot)$ in $\mathcal{F}^\varepsilon(q)$ and to $C^\varepsilon(q)$ in $\mathcal{S}^\varepsilon(q)$, that is

$$p^\varepsilon_{\partial\mathcal{S}^\varepsilon(q)}(q,\cdot) = \frac{\partial\psi^\varepsilon}{\partial n}(q,\cdot) \text{ on } \partial\mathcal{S}^\varepsilon(q).$$

Hence we obtain (5.39), by using (5.38) and the condition (5.14d) by using (5.65a). ☐

5.4.2. *Proof of Lemma 14.* Let $(\varepsilon, q) \in \mathfrak{Q}$ and $\mathfrak{p}_r(\varepsilon, q, \cdot) := \big(\mathsf{p}_{\partial\mathcal{S}_0,r}(\varepsilon, q, \cdot), p_{\partial\Omega,r}(\varepsilon, q, \cdot), C_r(\varepsilon, q)\big) \in F_0$ satisfying (5.53), that is

$$SL[\mathsf{p}_{\partial\mathcal{S}_0,r}(\varepsilon, q, \cdot)] + K_{\partial\mathcal{S}_0}(\varepsilon, q)[\mathsf{p}_{\partial\Omega,r}(\varepsilon, q, \cdot)] = g_{\partial\mathcal{S}_0}(\varepsilon, q, \cdot) \qquad \text{on } \partial\mathcal{S}_0, \tag{5.66a}$$

$$SL[\mathsf{p}_{\partial\Omega,r}(\varepsilon, q, \cdot)] + K_{\partial\Omega}(\varepsilon, q)[\mathsf{p}_{\partial\mathcal{S}_0,r}(\varepsilon, q, \cdot)] = g_{\partial\Omega}(\varepsilon, q, \cdot) \qquad \text{on } \partial\Omega, \tag{5.66b}$$

$$\int_{\partial\mathcal{S}_0} \mathsf{p}_{\partial\mathcal{S}_0,r}(\varepsilon, q, \cdot)\mathrm{d}s = 0. \tag{5.66c}$$

Let $\mathfrak{p}^\varepsilon(q, \cdot) = \big(p^\varepsilon_{\partial\mathcal{S}_0}(q, \cdot), p^\varepsilon_{\partial\Omega}(q, \cdot), C^\varepsilon(q)\big) \in F_0$ given by (5.54). In order to prove (5.36) we now verify the three parts of (5.64).

Verification of (5.64a)**.** Using again that the densities above are equal to the jumps of the normal derivatives of the associated single layer integrals and the conditions (2.16e), (2.16b), (5.17c) and (5.25c) we get that

$$\int_{\partial\mathcal{S}_0} \mathsf{p}^{-1}_{\cancel{\partial\Omega}}\mathrm{d}s = -1 \text{ and } \int_{\partial\mathcal{S}_0} \mathsf{p}^{0}_{\cancel{\partial\Omega}}(q, \cdot)\mathrm{d}s = \int_{\partial\mathcal{S}_0} \mathsf{p}^{1}_{\cancel{\partial\Omega}}(q, \cdot)\mathrm{d}s = 0. \tag{5.67}$$

As a consequence of (5.67) and of (5.66c) we get that the condition (5.64a) holds true.

Verification of (5.64b)**.** Using (2.16b), (5.40), (5.67) and (5.66c) we obtain, on $\partial\mathcal{S}_0$,

$$SL[p^\varepsilon_{\partial\mathcal{S}_0}(q, \cdot)] = C_{\cancel{\partial\Omega}} + \varepsilon\psi^0_{\cancel{\partial\Omega}}(q, \cdot) + \varepsilon^2\psi^1_{\cancel{\partial\Omega}}(q, \cdot) + \varepsilon^3 SL[p^\varepsilon_{\partial\mathcal{S}_0,r}(\varepsilon, q, \cdot)]. \tag{5.68}$$

Using Taylor's formula, (5.41) and (5.19) we get on $\partial\mathcal{S}_0$

(5.69)

$$\begin{aligned} K_{\partial\mathcal{S}_0}(\varepsilon, q)[p^\varepsilon_{\partial\Omega}(q, \cdot)](X) = {} & \psi^0_{\cancel{\partial\mathcal{S}}}(h, h) + \varepsilon D_x\psi^0_{\cancel{\partial\mathcal{S}}}(h, h)\cdot\big(R(\vartheta)X + \zeta_\vartheta\big) \\ & + \varepsilon^2\Big(\psi^2_{\cancel{\partial\mathcal{S}}}(q, h) + D_x\psi^1_{\cancel{\partial\mathcal{S}}}(q, h)\cdot R(\vartheta)X + \frac{1}{2}D^2_x\psi^0_{\cancel{\partial\mathcal{S}}}(h, h)\cdot\big(R(\vartheta)X, R(\vartheta)X\big)\Big) \\ & + \varepsilon^3\Big(K_{\partial\mathcal{S}_0}(\varepsilon, q)[\mathsf{p}_{\partial\Omega,r}(\varepsilon, q, \cdot)](X) - g_{\partial\mathcal{S}_0}(\varepsilon, q, X)\Big). \end{aligned}$$

We recall that the function $g_{\partial\mathcal{S}_0}(\varepsilon, q, \cdot)$ is defined in (5.49a).

Gathering (5.46), (5.54), (5.68) and (5.69) we obtain, on $\partial\mathcal{S}_0$,

$$\begin{aligned} -G(\varepsilon) + SL[p^\varepsilon_{\partial\mathcal{S}_0}(q, \cdot)] + K_{\partial\mathcal{S}_0}(\varepsilon, q)[p^\varepsilon_{\partial\Omega}(q, \cdot)] - C^\varepsilon(q) = {} & \varepsilon\big(\psi^0_{\cancel{\partial\Omega}}(q, \cdot) - P^0(q, \cdot)\big) \\ & + \varepsilon^2\big(\psi^1_{\cancel{\partial\Omega}}(q, \cdot) - P^1(q, \cdot)\big) \\ & + \varepsilon^3\Big(SL[p^\varepsilon_{\partial\mathcal{S}_0,r}(\varepsilon, q, \cdot)] + K_{\partial\mathcal{S}_0}(\varepsilon, q)[\mathsf{p}_{\partial\Omega,r}(\varepsilon, q, \cdot)] - g_{\partial\mathcal{S}_0}(\varepsilon, q, \cdot)\Big), \end{aligned}$$

where $P^0(q, \cdot)$ and $P^1(q, \cdot)$ are the harmonic polynomials defined respectively in (5.15) and in (5.22). Now taking into account the boundary conditions (5.17b) and (5.44b), and (5.66a) we get that (5.64b) holds true.

Remark 5. *Let us explain a bit how the ansatz of* $C^\varepsilon(q)$ *given by* (5.46) *and* (5.54) *was guessed. Taking into account* (5.68) *and* (5.69) *we multiply* $-G(\varepsilon) + SL[p^\varepsilon_{\partial\mathcal{S}_0}(q, \cdot)] + K_{\partial\mathcal{S}_0}(\varepsilon, q)[p^\varepsilon_{\partial\Omega}(q, \cdot)]$ *by* $-\mathsf{p}^{-1}_{\cancel{\partial\Omega}}$*, and integrate over* $\partial\mathcal{S}_0$*, combining with* (5.64b)*. We then simplify the resulting equation with the following observations:*

- *using* (2.15), (2.17) *and the first equality in* (5.63)*, we have:*

$$\int_{\partial\mathcal{S}_0} \mathsf{p}^{-1}_{\cancel{\partial\Omega}}(X)R(\vartheta)X\,\mathrm{d}s(X) = -\zeta_\vartheta, \tag{5.70}$$

so that

$$\int_{\partial\mathcal{S}_0} \mathsf{p}_{\cancel{\partial\Omega}}^{-1}(X) D_x\psi^0_{\cancel{\partial\mathcal{S}}}(h,h)\cdot R(\vartheta)X \,\mathrm{d}s(X) = -D_x\psi^0_{\cancel{\partial\mathcal{S}}}(h,h)\cdot\zeta_\vartheta,$$

$$\int_{\partial\mathcal{S}_0} \mathsf{p}_{\cancel{\partial\Omega}}^{-1}(X) D_x\psi^1_{\cancel{\partial\mathcal{S}}}(q,h)\cdot R(\vartheta)X \,\mathrm{d}s(X) = -D_x\psi^1_{\cancel{\partial\mathcal{S}}}(q,h)\cdot\zeta_\vartheta;$$

- *using again the first equality in* (5.63) *and* (5.23)*, we get:*

$$\int_{\partial\mathcal{S}_0} \mathsf{p}_{\cancel{\partial\Omega}}^{-1}(X) X^{\otimes 2}\,\mathrm{d}s(X) = T^2(\mathsf{p}_{\cancel{\partial\Omega}}^{-1}), \tag{5.71}$$

so that

$$\int_{\partial\mathcal{S}_0} \mathsf{p}_{\cancel{\partial\Omega}}^{-1}(X) D^2_x\psi^0_{\cancel{\partial\mathcal{S}}}(h,h)\cdot\big(R(\vartheta)X, R(\vartheta)X\big)\,\mathrm{d}s(X) = \langle R(\vartheta)^t\, D^2_x\psi^0_{\cancel{\partial\mathcal{S}}}(h,h)R(\vartheta), T^2(\mathsf{p}_{\cancel{\partial\Omega}}^{-1})\rangle_{\mathbb{R}^{2\times 2}};$$

- *using* (5.16), (5.17b), (5.24), (5.25b).

Then we deduce that if (5.64a) *and* (5.64b) *hold true then* $C^\varepsilon(q)$ *should be given by* (5.46) *and* (5.54) *with*

$$C_r(\varepsilon,q) := -\int_{\partial\mathcal{S}_0} \mathsf{p}_{\cancel{\partial\Omega}}^{-1}\big(SL[p^\varepsilon_{\partial\mathcal{S},r}(\varepsilon,q,\cdot)] + K_{\partial\mathcal{S}_0}(\varepsilon,q)[\mathsf{p}_{\partial\Omega,r}(\varepsilon,q,\cdot)] - g_{\partial\mathcal{S}_0}(\varepsilon,q,\cdot)\big)\mathrm{d}s.$$

We see that the ansatz (5.46)-(5.54) *leads to a reminder* $C_r(\varepsilon,q)$ *of order* $O(1)$*, which encourages the try of* (5.46) *as an approximate solution.*

Verification of (5.64c)**.** First, using (5.54), (5.41), we obtain, on $\partial\Omega$,

$$SL[p^\varepsilon_{\partial\Omega}(q,\cdot)] = \psi^0_{\cancel{\partial\mathcal{S}}}(h,\cdot) + \varepsilon\psi^1_{\cancel{\partial\mathcal{S}}}(q,\cdot) + \varepsilon^2\,\psi^2_{\cancel{\partial\mathcal{S}}}(q,\cdot) + \varepsilon^3\, SL[\mathsf{p}_{\partial\Omega,r}(\varepsilon,q,\cdot)]. \tag{5.72}$$

On the other hand, using Taylor's formula, (5.54), (5.67), (5.66c), (5.70), (5.43) (5.71) and (5.49b), we get, for $x\in\partial\Omega$,

$$\begin{aligned} K_{\partial\Omega}(\varepsilon,q)[p^\varepsilon_{\partial\mathcal{S}_0}(q,\cdot)](x) &= -G(x-h) + \varepsilon DG(x-h)\cdot\zeta_\vartheta \\ &+ \varepsilon^2\Big(-R(\vartheta)^t\, DG(x-h)\cdot T^1(\mathsf{p}^0_{\cancel{\partial\Omega}}(q,\cdot)) - \frac{1}{2}\langle R(\vartheta)^t\, D^2_x G(x-h)R(\vartheta), T^2(\mathsf{p}_{\cancel{\partial\Omega}}^{-1})\rangle_{\mathbb{R}^{2\times 2}}\Big) \\ &+ \varepsilon^3\Big(K_{\partial\Omega}(\varepsilon,q)[p^\varepsilon_{\partial\mathcal{S}_0,r}(\varepsilon,q,\cdot)](x) - g_{\partial\Omega}(\varepsilon,q,x)\Big). \end{aligned} \tag{5.73}$$

Gathering (5.72) and (5.73), the equation (5.64c) now reads, for $x\in\partial\Omega$,

$$\begin{aligned} SL[p^\varepsilon_{\partial\Omega}(q,\cdot)](x) + K_{\partial\Omega}(\varepsilon,q)[p^\varepsilon_{\partial\mathcal{S}_0}(q,\cdot)](x) &= \psi^0_{\cancel{\partial\mathcal{S}}}(h,x) - G(x-h) \\ &+ \varepsilon\Big(\psi^1_{\cancel{\partial\mathcal{S}}}(q,x) + DG(x-h)\cdot\zeta_\vartheta\Big) \\ &+ \varepsilon^2\Big(\psi^2_{\cancel{\partial\mathcal{S}}}(q,x) - Q^2(q,x)\Big) \\ &+ \varepsilon^3\Big(SL[\mathsf{p}_{\partial\Omega,r}(\varepsilon,q,\cdot)] + K_{\partial\Omega}(\varepsilon,q)[p^\varepsilon_{\partial\mathcal{S}_0,r}(\varepsilon,q,\cdot)](x) - g_{\partial\Omega}(\varepsilon,q,x)\Big), \end{aligned} \tag{5.74}$$

where $Q^2(q,x)$ denotes the harmonic polynomial defined in (5.42).

Taking now the boundary conditions (1.23), (5.18b), (5.44b) and (5.66b) into account, we deduce from the equation (5.74) that (5.64c) holds true. □

5.4.3. *Proof of Lemma 16.* It is straightforward to see that for any $(\varepsilon, q) \in \mathfrak{Q}$, $A(\varepsilon, q)$ is linear continuous. Let (ε, q) in $\mathcal{Q}$, with $q = (\vartheta, h) \in \mathbb{R} \times \Omega$. Let us introduce, for any $\mathfrak{p} := (\mathrm{p}_{\mathcal{S}_0}, \mathrm{p}_{\partial\Omega}, C) \in F_{-\frac{1}{2}}$,

$$\begin{aligned} L[\mathfrak{p}] &:= (SL[\mathrm{p}_{\mathcal{S}_0}], SL[\mathrm{p}_{\partial\Omega}], C), \\ K(h)[\mathfrak{p}] &:= (\tilde{K}_{\partial\mathcal{S}_0}(h)[\mathrm{p}_{\partial\Omega}] - C, 0, \int_{\partial\mathcal{S}_0} \mathrm{p}_{\partial\mathcal{S}_0} \mathrm{d}s - C), \\ T(\varepsilon, q)[\mathfrak{p}] &:= (T_{\partial\mathcal{S}}(\varepsilon, q)[\mathrm{p}_{\partial\Omega}], T_{\partial\Omega}(\varepsilon, q)[\mathrm{p}_{\partial\mathcal{S}_0}], 0), \end{aligned}$$

so that we can write A in the following form: on F_0,

$$A(\varepsilon, q) = L + K(h) + \varepsilon T(\varepsilon, q). \tag{5.75}$$

We first consider the operator $L + K(h)$. According to (5.3), the operator L is Fredholm with index zero and since for each $h \in \overline{\Omega_\delta}$, $K(h)$ is compact, we deduce that $L + K(h)$ is Fredholm with index zero. It follows that in order to prove that $L + K(h)$ is an isomorphism, it is sufficient to prove that its kernel is trivial.

Consider $\mathfrak{p} := (\mathrm{p}_{\partial\mathcal{S}_0}, \mathrm{p}_{\partial\Omega}, C) \in F_{-\frac{1}{2}}$ such that $\big(L + K(h)\big)[\mathfrak{p}] = 0$. Since the logarithmic capacity $\mathrm{Cap}_{\partial\Omega}$ of $\partial\Omega$ satisfies $\mathrm{Cap}_{\partial\Omega} \neq 1$, according to (5.5), the second equation $SL[\mathrm{p}_{\partial\Omega}] = 0$ implies $\mathrm{p}_{\partial\Omega} = 0$. Then reporting in the first equation, we get $SL[\mathrm{p}_{\mathcal{S}_0}] = C$, whereas the third equation reads $\int_{\partial\mathcal{S}_0} \mathrm{p}_{\partial\mathcal{S}_0} \mathrm{d}s = 0$. Thus according to (5.4), we obtain $\mathrm{p}_{\partial\mathcal{S}_0} = 0$ and thus $C = 0$. This proves that the kernel of $L + K(h)$ is trivial, and consequently that for any $h \in \Omega_\delta$, $L + K(h)$ is an isomorphism.

Now using that the dependence of K on h is Lipschitz, we deduce that $L + K(h)$ has locally a bounded inverse. By compactness of $\overline{\Omega_\delta}$, it follows that $L + K(h)$ has a bounded inverse for h running over $\overline{\Omega_\delta}$.

Since the operators $(T^\varepsilon)_{\varepsilon \in (0,1)}$ are bounded in the space of bounded operators from F_0 to F_1 we can then easily deduce the result from (5.75). □

5.4.4. *Proof of Lemma 17.* The proof of Lemma 17 relies on Lemma 12. First we use Lemma 12 with $\mathcal{C} = \partial\Omega$, $b = G$ and $\mathrm{p}_{\mathcal{C}} = \mathrm{p}_{\partial\Omega}$ to obtain that $\tilde{K}_{\partial\mathcal{S}_0}$ satisfies (5.56). Next we apply Lemma 12, (i) for any $(\varepsilon, q) \in \mathfrak{Q}_{\delta,\varepsilon_0}$, with $\mathcal{C} = \partial\Omega$, $b(x,y) = \eta_1(\varepsilon, q, x, y)$ and $\mathrm{p}_{\mathcal{C}} = \mathrm{p}_{\partial\Omega}$ and with $\mathcal{C} = \partial\mathcal{S}_0$, $b(x,y) = \eta_1(\varepsilon, \vartheta, x, -y, h)$ and $\mathrm{p}_{\mathcal{C}} = \mathrm{p}_{\mathcal{S}_0}$ to get that $T_{\partial\mathcal{S}_0}(\varepsilon, q)$ and $T_{\partial\Omega}(\varepsilon, q)$ satisfy (5.57). □

5.4.5. *Proof of Lemma 18.* Let $\delta > 0$. Let us first observe that for any $(\varepsilon, q) \in \mathfrak{Q}_\delta$, for any $\mathfrak{p} := (\mathrm{p}_{\partial\mathcal{S}_0}, \mathrm{p}_{\partial\Omega}, C) \in F_0$ satisfying the condition

$$\int_{\partial\mathcal{S}_0} \mathrm{p}_{\partial\mathcal{S}_0} \mathrm{d}s = 0, \tag{5.76}$$

one has

$$\mathfrak{A}(\varepsilon, q)[\mathfrak{p}] = A(\varepsilon, q)[\mathfrak{p}].$$

Indeed

$$\tilde{K}_{\partial\mathcal{S}_0}(h) = K_{\partial\mathcal{S}_0}(0, 0, h)[\mathrm{p}_{\partial\Omega}] = SL[\mathrm{p}_{\partial\Omega}](h)$$

and first order Taylor expansions yield that

$$\begin{aligned} K_{\partial\mathcal{S}_0}(\varepsilon, q)[\mathrm{p}_{\partial\Omega}] - \tilde{K}_{\partial\mathcal{S}_0}(h)[\mathrm{p}_{\partial\Omega}] &= \varepsilon T_{\partial\mathcal{S}_0}(\varepsilon, q)[\mathrm{p}_{\partial\Omega}], \\ K_{\partial\Omega}(\varepsilon, q)[\mathrm{p}_{\partial\mathcal{S}_0}] &= \varepsilon T_{\partial\Omega}(\varepsilon, q)[\mathrm{p}_{\partial\mathcal{S}_0}]. \end{aligned}$$

We emphasize in particular that the last equality relies on the condition (5.76).

Now, consider $\varepsilon_0 \in (0,1)$ obtained by applying Lemma 16. For any $(\varepsilon, q) \in \mathfrak{Q}_{\delta,\varepsilon_0}$, consider $\mathfrak{p}_r(\varepsilon, q, \cdot) = \big(\mathrm{p}_{\partial\mathcal{S}_0,r}(\varepsilon, q, \cdot), p_{\partial\Omega,r}(\varepsilon, q, \cdot), C_r(\varepsilon, q)\big)$ given by (5.61) . It belongs to F_0 and satisfies (5.66c) and consequently

$$\mathfrak{A}(\varepsilon, q)[p_r(\varepsilon, q, \cdot)] = A(\varepsilon, q)[p_r(\varepsilon, q, \cdot)] = \mathfrak{g}(\varepsilon, q, \cdot).$$

Moreover we have the estimate

$$\|p_r(\varepsilon, q, \cdot)\|_{F_0} \leqslant \|A(\varepsilon, q)^{-1}\|_{\mathcal{L}(F_1;F_0)} \|\mathfrak{g}(\varepsilon, q, \cdot)\|_{F_1}.$$

The estimates (5.52) and (5.60) entail that $\mathfrak{p}_r$ is in $L^\infty(Q_{\delta,\varepsilon_0}; F_0)$, which concludes the proof. □

5.5. **Proof of Proposition 8.** In this subsection, we establish Proposition 8 by following the lines of the proof of Proposition 7. A first step consists in transforming the (Neumann) problem defining the Kirchhoff potentials into a Dirichlet one, so that we can more closely follow the steps of Subsection 5.3.

5.5.1. *Reduction to a Dirichlet problem.* We consider the functions $\overline{\varphi}_j^\varepsilon(q, \cdot)$, for $j = 1, 2, 3$, as the solution to the following Dirichlet boundary value problem in $\mathcal{F}^\varepsilon(q)$:

$$-\Delta\overline{\varphi}_j^\varepsilon(q, \cdot) = 0 \quad \text{in } \mathcal{F}^\varepsilon(q), \tag{5.77a}$$

$$\overline{\varphi}_j^\varepsilon(q, \cdot) = \overline{K}_j(q, \cdot) + c_j^\varepsilon(q) \quad \text{on } \partial\mathcal{S}^\varepsilon(q), \tag{5.77b}$$

$$\overline{\varphi}_j^\varepsilon(q, \cdot) = 0 \quad \text{on } \partial\Omega, \tag{5.77c}$$

where the functions $\overline{K}_j(q, \cdot)$ are given by

$$\overline{K}_j(q, \cdot) := \begin{cases} \frac{1}{2}|x-h|^2 & \text{if} \quad j=1, \\ -R(\vartheta)^t\,(x-h)\cdot e_2 & \text{if} \quad j=2, \\ R(\vartheta)^t\,(x-h)\cdot e_1 & \text{if} \quad j=3, \end{cases}$$

and the constants $c_j^\varepsilon(q)$ are such that:

$$\int_{\partial\mathcal{S}^\varepsilon(q)} \frac{\partial\overline{\varphi}_j^\varepsilon}{\partial n}(q, \cdot)\mathrm{d}s = 0. \tag{5.77d}$$

Let us recall that e_1 and e_2 are the unit vectors of the canonical basis.

The constants $c_j^\varepsilon(q)$ are determined by

$$c_j^\varepsilon(q) = C^\varepsilon(q) \int_{\partial\mathcal{S}^\varepsilon(q)} \frac{\partial\overline{\phi}_j^\varepsilon}{\partial n}(q, \cdot)\mathrm{d}s, \tag{5.78}$$

where $\overline{\phi}_j^\varepsilon$, $j = 1, 2, 3$ are the solutions of

$$-\Delta\overline{\phi}_j^\varepsilon(q, \cdot) = 0 \quad \text{in } \mathcal{F}^\varepsilon(q), \tag{5.79}$$

$$\overline{\phi}_j^\varepsilon(q, \cdot) = \overline{K}_j(q, \cdot) \quad \text{on } \partial\mathcal{S}^\varepsilon(q), \tag{5.80}$$

$$\overline{\phi}_j^\varepsilon(q, \cdot) = 0 \quad \text{on } \partial\Omega. \tag{5.81}$$

We will use the vector notation:

$$\overline{\boldsymbol{\varphi}}^\varepsilon := (\overline{\varphi}_1^\varepsilon, \overline{\varphi}_2^\varepsilon, \overline{\varphi}_3^\varepsilon)^t. \tag{5.82}$$

The functions $\overline{\varphi}_j^\varepsilon(q, \cdot)$ are harmonically conjugated to the Kirchhoff's potentials $\varphi_j^\varepsilon(q, \cdot)$, up to a rotation, as shown in the following result.

Lemma 19. *For any $(\varepsilon, q) \in \mathfrak{Q}$, with $q = (\vartheta, h)$, there holds in $\mathcal{F}^\varepsilon(q)$,*

$$\nabla\varphi_j^\varepsilon(q, \cdot) = \nabla^\perp\check{\varphi}_j^\varepsilon(q, \cdot), \tag{5.83}$$

where

$$\Big(\check{\varphi}_1^\varepsilon(q, \cdot),\, \check{\varphi}_2^\varepsilon(q, \cdot),\, \check{\varphi}_3^\varepsilon(q, \cdot)\Big) := \mathcal{R}(\vartheta)\Big(\overline{\varphi}_1^\varepsilon(q, \cdot),\, \overline{\varphi}_2^\varepsilon(q, \cdot),\, \overline{\varphi}_3^\varepsilon(q, \cdot)\Big). \tag{5.84}$$

Proof of Lemma 19. First, let us recall that for any $(\varepsilon, q) \in \mathfrak{Q}$, the system

$$\operatorname{div} u = 0 \quad \text{in } \mathcal{F}^\varepsilon(q), \tag{5.85a}$$

$$\operatorname{curl} u = 0 \quad \text{in } \mathcal{F}^\varepsilon(q), \tag{5.85b}$$

$$u \cdot n = 0 \quad \text{on } \partial\Omega, \tag{5.85c}$$

$$u \cdot n = K_j(q, \cdot) \quad \text{on } \partial\mathcal{S}^\varepsilon(q), \tag{5.85d}$$

$$\int_{\partial\mathcal{S}^\varepsilon(q)} u \cdot \tau \mathrm{d}s = 0, \tag{5.85e}$$

has a unique solution u, say in $H^1(\mathcal{F}^\varepsilon(q))$. Then one observes that both $\nabla \varphi_j^\varepsilon(q, \cdot)$ and $\nabla^\perp \check{\varphi}_j^\varepsilon(q, \cdot)$ solve (5.85). In particular let us emphasize that, on $\partial\mathcal{S}^\varepsilon(q)$,

$$\Big(n \cdot \nabla^\perp \overline{\varphi}_j^\varepsilon(q, \cdot)\Big)_{j=1,2,3} = \Big(\frac{\partial \overline{K}_j}{\partial \tau}(q, \cdot)\Big)_{j=1,2,3} = \mathcal{R}(\vartheta)^t \boldsymbol{K}(q, \cdot),$$

so that, for $j = 1, 2, 3$, $\nabla^\perp \check{\varphi}_j^\varepsilon(q, \cdot)$ satisfies (5.85d), and the condition (5.77d) ensures that (5.85e) is satisfied. $\square$

In the case without exterior boundary we consider in the same way $\overline{\varphi}_{\cancel{\partial\Omega},j}$ as the solution of

$$-\Delta \overline{\varphi}_{\cancel{\partial\Omega},j} = 0 \quad \text{in } \mathbb{R}^2 \setminus \mathcal{S}_0, \tag{5.86a}$$

$$\overline{\varphi}_{\cancel{\partial\Omega},j}(\cdot) = \overline{K}_j(0, \cdot) + c_{\cancel{\partial\Omega},j} \quad \text{on } \partial\mathcal{S}_0, \tag{5.86b}$$

$$\overline{\varphi}_{\cancel{\partial\Omega},j}(x) \to 0 \quad \text{as } |x| \to +\infty, \tag{5.86c}$$

where the constant $c_{\cancel{\partial\Omega},j}$ is such that

$$\int_{\partial\mathcal{S}_0} \frac{\partial \overline{\varphi}_{\cancel{\partial\Omega},j}}{\partial n} \mathrm{d}s = 0. \tag{5.86d}$$

The existence and uniqueness of such a constant $c_{\cancel{\partial\Omega},j}$ is provided by a similar argument as for (5.78)-(5.79). Proceeding as in the proof of Lemma 19 we get

$$\nabla \varphi_{\cancel{\partial\Omega},j} = \nabla^\perp \overline{\varphi}_{\cancel{\partial\Omega},j}, \tag{5.87}$$

where the functions $\varphi_{\cancel{\partial\Omega},j}$, for $j = 1, 2, 3$, are the Kirchhoff's potentials in $\mathbb{R}^2 \setminus \mathcal{S}_0$ defined in (2.8). As before we introduce the vector notation for the functions $\overline{\varphi}_{\cancel{\partial\Omega},j}$:

$$\overline{\boldsymbol{\varphi}}_{\cancel{\partial\Omega}} := (\overline{\varphi}_{\cancel{\partial\Omega},1}, \overline{\varphi}_{\cancel{\partial\Omega},2}, \overline{\varphi}_{\cancel{\partial\Omega},3}). \tag{5.88}$$

Then, following the strategy of Proposition 7 we will establish the following result.

Proposition 9. *Let $\delta > 0$. There exists $\varepsilon_0 \in (0, 1)$ and*

(i) there exists $\boldsymbol{p}_{\partial\mathcal{S}_0,r} \in L^\infty\big(\mathfrak{Q}_{\delta,\varepsilon_0}; L^2(\partial\mathcal{S}_0; \mathbb{R}^3)\big)$ such that for any $(\varepsilon, q) \in \mathfrak{Q}_{\delta,\varepsilon_0}$, with $q = (\vartheta, h)$, for any $X \in \partial\mathcal{S}_0$,

$$\frac{\partial \overline{\boldsymbol{\varphi}}^\varepsilon}{\partial n}(q, \varepsilon R(\vartheta)X + h) = I_\varepsilon\Big(\frac{\partial \overline{\boldsymbol{\varphi}}_{\cancel{\partial\Omega}}}{\partial n}(X) + \varepsilon^2 \boldsymbol{p}_{\partial\mathcal{S}_0,r}(\varepsilon, q, X)\Big), \tag{5.89}$$

(ii) there exists $\boldsymbol{p}_{\partial\Omega,r} \in L^\infty\big(\mathfrak{Q}_{\delta,\varepsilon_0}; L^2(\partial\Omega; \mathbb{R}^3)\big)$ such that for any $(\varepsilon, q) \in \mathfrak{Q}_{\delta,\varepsilon_0}$, for any $x \in \partial\Omega$,

$$\frac{\partial \overline{\boldsymbol{\varphi}}^\varepsilon}{\partial n}(q, x) = I_\varepsilon \, \varepsilon^2 \, \boldsymbol{p}_{\partial\Omega,r}(\varepsilon, q, x). \tag{5.90}$$

Once Proposition 9 is obtained, Proposition 8 follows by using additionally Lemma 19 and (5.87). $\square$

Hence the rest of Subsection 5.5 is devoted to the proof of Proposition 9.

5.5.2. *Proof of Proposition 9.* We will follow the lines of the proof of Proposition 7 and will therefore highlight the differences.

Let $j = 1, 2$ or 3. We introduce the solution κ_j^ε of the interior Dirichlet problem

$$\Delta \kappa_j^\varepsilon(q,\cdot) = 0 \text{ in } \mathcal{S}^\varepsilon(q), \quad \kappa_j^\varepsilon(q,\cdot) = \overline{K}_j(q,\cdot) \text{ on } \partial\mathcal{S}^\varepsilon(q), \tag{5.91}$$

which merely gives $\kappa_j^\varepsilon(q,\cdot) = \overline{K}_j(q,\cdot)$ in $\mathcal{S}^\varepsilon(q)$ in the cases $j = 2$ or 3.

First step. Reduction to an integral equation. As a counterpart of Lemma 13 we have the following result.

Lemma 20. *For any $(\varepsilon, q) \in \mathfrak{Q}$, let $\mathfrak{p}_j^\varepsilon(q,\cdot) = \big(p^\varepsilon_{\partial\mathcal{S}_0,j}(q,\cdot), p^\varepsilon_{\partial\Omega,j}(q,\cdot), C_j^\varepsilon(q)\big) \in F_0$ such that*

$$\mathfrak{A}(\varepsilon,q)\big[\mathfrak{p}_j^\varepsilon(q,\cdot)\big] = (\varepsilon\overline{K}_j(0,\cdot), 0, 0). \tag{5.92}$$

Then the function

$$\overline{\varphi}_j^\varepsilon(q,\cdot) := \varepsilon^{\delta_{1,j}}\Big(SL[p^\varepsilon_{\partial\mathcal{S}^\varepsilon(q),j}(q,\cdot)] + SL[p^\varepsilon_{\partial\Omega,j}(q,\cdot)]\Big), \tag{5.93}$$

where the density $p^\varepsilon_{\partial\mathcal{S}^\varepsilon(q),j}(q,\cdot)$ on $\partial\mathcal{S}^\varepsilon(q)$ is defined through the following relation:

$$\text{for } X \in \partial\mathcal{S}_0, \quad p^\varepsilon_{\partial\mathcal{S}_0,j}(q,X) := \varepsilon p^\varepsilon_{\partial\mathcal{S}^\varepsilon(q),j}(q, \varepsilon R(\vartheta)X + h), \text{ with } q = (\vartheta, h), \tag{5.94}$$

is the solution of (5.77) *with*

$$c_j^\varepsilon(q) = \varepsilon^{\delta_{1,j}} C_j^\varepsilon(q). \tag{5.95}$$

Moreover the normal derivative $\frac{\partial\overline{\varphi}_j^\varepsilon}{\partial n}$ on $\partial\mathcal{S}^\varepsilon(q)$ and on $\partial\Omega$ is given respectively by the formula:

$$\text{for } X \in \partial\mathcal{S}_0, \quad \frac{\partial\overline{\varphi}_j^\varepsilon}{\partial n}(q, \varepsilon R(\vartheta)X + h) = \varepsilon^{\delta_{1,j}-1}\Big(p^\varepsilon_{\partial\mathcal{S}_0,j}(q,X) + \varepsilon\frac{\partial\kappa_j^1}{\partial n}(0,X)\Big), \tag{5.96}$$

$$\text{for } x \in \partial\Omega, \quad \frac{\partial\overline{\varphi}_j^\varepsilon}{\partial n}(q,x) = \varepsilon^{\delta_{1,j}}\, p^\varepsilon_{\partial\Omega,j}(q,x). \tag{5.97}$$

Above $\delta_{1,j}$ stands for the standard Kronecker symbol.

Proof of Lemma 20. For any densities $p^\varepsilon_{\partial\mathcal{S}^\varepsilon(q),j}(q,\cdot) \in H^{-\frac{1}{2}}(\partial\mathcal{S}^\varepsilon(q))$ and $p^\varepsilon_{\partial\Omega,j}(q,\cdot) \in H^{-\frac{1}{2}}(\partial\Omega)$, the right hand side of (5.93) is in $H^1_{loc}(\mathbb{R}^2)$ and harmonic in $\mathcal{F}^\varepsilon(q)$ and in $\mathbb{R}^2 \setminus \mathcal{F}^\varepsilon(q)$. In particular the equation (5.77a) is satisfied when $\overline{\varphi}_j^\varepsilon(q,\cdot)$ is given by (5.93).

Next we observe that (5.92) is equivalent to:

$$\int_{\partial\mathcal{S}_0} p^\varepsilon_{\partial\mathcal{S}_0,j}(q,\cdot)\mathrm{d}s = 0, \tag{5.98a}$$

$$SL[p^\varepsilon_{\partial\mathcal{S}_0,j}(q,\cdot)] + K_{\partial\mathcal{S}_0}(\varepsilon,q)[p^\varepsilon_{\partial\Omega,j}(q,\cdot)] = \varepsilon\overline{K}_j(0,\cdot) + C_j^\varepsilon(q), \text{ on } \partial\mathcal{S}_0, \tag{5.98b}$$

$$SL[p^\varepsilon_{\partial\Omega,j}(q,\cdot)] + K_{\partial\Omega}(\varepsilon,q)[p^\varepsilon_{\partial\mathcal{S}_0,j}(q,\cdot)] = 0, \text{ on } \partial\Omega. \tag{5.98c}$$

Thanks to a change of variable, and using

$$\overline{K}_j(q, \varepsilon R(\vartheta)X + h) = \varepsilon^{1+\delta_{1,j}}\,\overline{K}_j(0,X), \tag{5.99}$$

this can be recast as

$$\int_{\partial\mathcal{S}^\varepsilon(q)} p^\varepsilon_{\partial\mathcal{S}^\varepsilon(q),j}(q,\cdot)\mathrm{d}s = 0, \tag{5.100a}$$

$$SL[p^\varepsilon_{\partial\mathcal{S}^\varepsilon(q),j}(q,\cdot)] + SL[p^\varepsilon_{\partial\Omega,j}(q,\cdot)] = \overline{K}_j(q,\cdot) + c_j^\varepsilon(q) \quad \text{on } \partial\mathcal{S}^\varepsilon(q), \tag{5.100b}$$

$$SL[p^\varepsilon_{\partial\Omega,j}(q,\cdot)] + SL[p^\varepsilon_{\partial\mathcal{S}^\varepsilon(q),j}(q,\cdot)] = 0 \quad \text{on } \partial\Omega. \tag{5.100c}$$

In particular we infer from the two last equations that the boundary conditions (5.77b) and (5.77c) are satisfied when $\overline{\varphi}_j^\varepsilon(q,\cdot)$ is given by (5.93).

Moreover the single-layer potential $SL[p^\varepsilon_{\partial\Omega,j}(q,\cdot)]$ is harmonic in a neighborhood of $\partial\mathcal{S}^\varepsilon(q)$ so that, according to (5.2), when $\overline{\varphi}_j^\varepsilon(q,\cdot)$ is given by (5.93), the density $\varepsilon^{\delta_{1,j}} p^\varepsilon_{\partial\mathcal{S}^\varepsilon(q),j}(q,\cdot)$ is equal to the jump across $\partial\mathcal{S}^\varepsilon(q)$ of the normal derivatives of the function equal to $\overline{\varphi}_j^\varepsilon(q,\cdot)$ in $\mathcal{F}^\varepsilon(q)$ and to $\kappa_j^\varepsilon(q,\cdot)$ in $\mathcal{S}^\varepsilon(q)$, that is

$$\varepsilon^{\delta_{1,j}} p^\varepsilon_{\partial\mathcal{S}^\varepsilon(q),j}(q,\cdot) = \frac{\partial\overline{\varphi}_j^\varepsilon}{\partial n}(q,\cdot) - \frac{\partial\kappa_j^\varepsilon}{\partial n}(q,\cdot) \text{ on } \partial\mathcal{S}^\varepsilon(q). \tag{5.101}$$

Using (5.91) and (5.99) we get, for any $X \in \mathcal{S}_0$,

$$\kappa_j^\varepsilon(q, \varepsilon R(\vartheta)X + h) = \varepsilon^{1+\delta_{1,j}}\, \kappa_j^1(0,X), \tag{5.102}$$

so that using (5.38), we get (5.96). A similar reasoning yields the formula (5.97).

Finally from (5.101) we deduce that, when $\overline{\varphi}_j^\varepsilon(q,\cdot)$ is given by (5.93), the condition (5.100a) entails the condition (5.77d). □

Second step. Construction of an approximate solution. Next we look for some solutions of (5.92) close to

$$\mathfrak{p}_{j,\text{app}}(\varepsilon,q,\cdot) = \big(\mathsf{p}_{\partial\mathcal{S}_0,j,\text{app}}(\varepsilon,q,\cdot), \mathsf{p}_{\partial\Omega,j,\text{app}}(\varepsilon,q,\cdot), C_{j,\text{app}}(\varepsilon,q)\big), \tag{5.103}$$

with

$$\mathsf{p}_{\partial\mathcal{S}_0,j,\text{app}}(\varepsilon,q,\cdot) = \varepsilon\mathsf{p}_{\cancel{\partial\Omega},j}(\cdot), \tag{5.104a}$$

$$\mathsf{p}_{\partial\Omega,j,\text{app}}(\varepsilon,q,\cdot) = \varepsilon^2\mathsf{p}_{\cancel{\partial\mathcal{S}},j}(q,\cdot), \tag{5.104b}$$

$$C_{j,\text{app}}(\varepsilon,q) = \varepsilon\mathsf{C}_j^1 + \varepsilon^2\mathsf{C}_j^2(q), \tag{5.104c}$$

where $\mathsf{p}_{\cancel{\partial\Omega},j}$, $\mathsf{p}_{\cancel{\partial\mathcal{S}},j}(q,\cdot)$, $\mathsf{C}_j^1(q)$ and $\mathsf{C}_j^2(q)$ are defined as follows:

- *Definition of* $\mathsf{p}_{\cancel{\partial\Omega},j}$. We define $\mathsf{p}_{\cancel{\partial\Omega},j}$ as the density associated with the function $\overline{\varphi}_{\cancel{\partial\Omega},j}$ extended by $\overline{K}_j(0,\cdot) + c_{\cancel{\partial\Omega},j}$ in $\mathcal{S}_0$.

- *Definition of* $\mathsf{p}_{\cancel{\partial\mathcal{S}},j}(q,\cdot)$. Let $\overline{\varphi}_{\cancel{\partial\mathcal{S}_0},j}(q,\cdot)$ be the solution of
$$\Delta\overline{\varphi}_{\cancel{\partial\mathcal{S}},j}(q,\cdot) = 0 \text{ in } \Omega, \quad \overline{\varphi}_{\cancel{\partial\mathcal{S}},j}(q,\cdot) = R(\vartheta)^t\, DG(x-h)\cdot T^1(\mathsf{p}_{\cancel{\partial\Omega},j}) \text{ on } \partial\Omega,$$
where
$$T^1(\mathsf{p}_{\cancel{\partial\Omega},j}) := \int_{\partial\mathcal{S}_0} Y\,\mathsf{p}_{\cancel{\partial\Omega},j}(Y)\mathrm{d}s(Y).$$
Then we define $\mathsf{p}_{\cancel{\partial\mathcal{S}_0},j}(q,\cdot)$ as the density associated with the function $\overline{\varphi}_{\cancel{\partial\mathcal{S}},j}(q,\cdot)$ extended by $R(\vartheta)^t\, DG(x-h)\cdot T^1(\mathsf{p}_{\cancel{\partial\Omega},j}(q,\cdot))$ in $\mathbb{R}^2\setminus\Omega$.

- *Definition of* C_j^1 *and* $\mathsf{C}_j^2(q)$. We set:

$$\mathsf{C}_j^1 = c_{\cancel{\partial\Omega},j}, \tag{5.105}$$

$$\mathsf{C}_j^2(q) = \overline{\varphi}_{\cancel{\partial\mathcal{S}},j}(q,h). \tag{5.106}$$

Now we look for a solution $\mathfrak{p}_j^\varepsilon(q,\cdot)$ to the equation (5.92) in the form

$$\mathfrak{p}_j^\varepsilon(q,\cdot) = \mathfrak{p}_{\text{app},j}(\varepsilon,q,\cdot) + \varepsilon^3\,\mathfrak{p}_{r,j}(\varepsilon,q,\cdot). \tag{5.107}$$

Hence the goal in this step of the proof is to reduce the equation (5.92) to an equation for the reminder $\mathfrak{p}_{r,j}(\varepsilon,q,\cdot)$ with some source terms depending on $\mathfrak{p}_{\mathrm{app},j}$. Let $\Big(g_{\partial\mathcal{S}_0}(\varepsilon,q,\cdot),g_{\partial\Omega}(\varepsilon,q,\cdot)\Big)\in H^1(\partial\mathcal{S}_0)\times H^1(\partial\Omega)$ defined by

$$-g_{\partial\mathcal{S}_0,j}(\varepsilon,q,\cdot):=\int_{\partial\Omega}\mathsf{p}_{\cancel{\partial\mathcal{S}},j}(q,y)\eta_1(\varepsilon,q,\cdot,y)\mathrm{d}s(y), \tag{5.108a}$$

$$-g_{\partial\Omega,j}(\varepsilon,q,\cdot):=\int_{\partial\mathcal{S}_0}\mathsf{p}_{\cancel{\partial\Omega},j}(y)\eta_1(\varepsilon,\vartheta,\cdot,-y,h)\mathrm{d}s(y), \tag{5.108b}$$

We recall that the functions η_N, for $N\geqslant 1$, are defined in (5.50). Let

$$\mathfrak{g}_j(\varepsilon,q,\cdot):=\big(g_{\partial\mathcal{S}_0,j}(\varepsilon,q,\cdot),g_{\partial\Omega,j}(\varepsilon,q,\cdot),0\big). \tag{5.109}$$

Applying Lemma 12, (ii), we get that

$$\mathfrak{g}_j\text{ is in }L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0};\tilde{F}_1), \tag{5.110}$$

where we recall that $\tilde{F}_1$ was defined in (5.55). Now we can establish the following.

Lemma 21. *For any $(\varepsilon,q)\in\mathfrak{Q}$, let $\mathfrak{p}_{r,j}(\varepsilon,q,\cdot)\in F_0$ satisfying:*

$$\mathfrak{A}(\varepsilon,q)[\mathfrak{p}_{r,j}(\varepsilon,q,\cdot)]=\mathfrak{g}_j(\varepsilon,q,\cdot). \tag{5.111}$$

Then $\mathfrak{p}^\varepsilon_j(q,\cdot)$ given by (5.107) *is solution of* (5.92).

Proof of Lemma 21. Let $(\varepsilon,q)\in\mathfrak{Q}$ and $\mathfrak{p}_{r,j}(\varepsilon,q,\cdot):=\big(\mathsf{p}_{\partial\mathcal{S}_0,r,j}(\varepsilon,q,\cdot),p_{\partial\Omega,r,j}(\varepsilon,q,\cdot),C_{r,j}(\varepsilon,q)\big)\in F_0$ satisfying (5.111), that is

$$SL[\mathsf{p}_{\partial\mathcal{S}_0,r,j}(\varepsilon,q,\cdot)]+K_{\partial\mathcal{S}_0}(\varepsilon,q)[\mathsf{p}_{\partial\Omega,r,j}(\varepsilon,q,\cdot)]=g_{\partial\mathcal{S}_0,j}(\varepsilon,q,\cdot), \tag{5.112a}$$

$$SL[\mathsf{p}_{\partial\Omega,r}(\varepsilon,q,\cdot)]+K_{\partial\Omega}(\varepsilon,q)[\mathsf{p}_{\partial\mathcal{S}_0,r,j}(\varepsilon,q,\cdot)]=g_{\partial\Omega,j}(\varepsilon,q,\cdot), \tag{5.112b}$$

$$\int_{\partial\mathcal{S}_0}\mathsf{p}_{\partial\mathcal{S}_0,r}(\varepsilon,q,\cdot)\mathrm{d}s=0. \tag{5.112c}$$

Let $\mathfrak{p}^\varepsilon_j(q,\cdot):=\big(p^\varepsilon_{\partial\mathcal{S}_0,j}(q,\cdot),p^\varepsilon_{\partial\Omega,j}(q,\cdot),C^\varepsilon_j(q)\big)\in F_0$ given by (5.107). We now check (5.98) which is the detailed version of (5.92) .

Proof of (5.98a). By definition of $\mathsf{p}_{\cancel{\partial\Omega},j}$ we get that

$$\int_{\partial\mathcal{S}_0}\mathsf{p}_{\cancel{\partial\Omega},j}\,\mathrm{d}s=0. \tag{5.113}$$

As a consequence of (5.113) and of (5.112c) we get that the condition (5.98a) holds true.

Proof of (5.98b). On $\partial\mathcal{S}_0$, we have on the one hand:

$$SL[p^\varepsilon_{\partial\mathcal{S}_0,j}(q,\cdot)]=\varepsilon\overline{\varphi}_{\cancel{\partial\Omega},j}+\varepsilon^3 SL[p^\varepsilon_{\partial\mathcal{S}_0,r,j}(\varepsilon,q,\cdot)], \tag{5.114}$$

and, using Taylor's formula, on the other hand:

$$K_{\partial\mathcal{S}_0}(\varepsilon,q)[p^\varepsilon_{\partial\Omega,j}]=\varepsilon^2\overline{\varphi}_{\cancel{\partial\mathcal{S}},j}(q,h)+\varepsilon^3\Big(K_{\partial\mathcal{S}_0}(\varepsilon,q)[\mathsf{p}_{\partial\Omega,r,j}(\varepsilon,q,\cdot)]-g_{\partial\mathcal{S}_0,j}(\varepsilon,q,\cdot)\Big). \tag{5.115}$$

Gathering this, we obtain, on $\partial\mathcal{S}_0$,

$$\begin{aligned}SL[p^\varepsilon_{\partial\mathcal{S}_0,j}(q,\cdot)]&+K_{\partial\mathcal{S}_0}(\varepsilon,q)[p^\varepsilon_{\partial\Omega,j}(q,\cdot)]-\varepsilon\overline{K}_j(0,\cdot)-C^\varepsilon_j(q)\\&=\varepsilon\Big(\overline{\varphi}_{\cancel{\partial\Omega},j}(q,\cdot)-\overline{K}_j(0,\cdot)-\mathsf{C}^1_j\Big)+\varepsilon^2\Big(\overline{\varphi}_{\cancel{\partial\mathcal{S}},j}(q,h)-\mathsf{C}^2_j(q)\Big)\\&\quad+\varepsilon^3\Big(SL[p^\varepsilon_{\partial\mathcal{S}_0,r,j}(\varepsilon,q,\cdot)]+K_{\partial\mathcal{S}_0}(\varepsilon,q)[\mathsf{p}_{\partial\Omega,r,j}(\varepsilon,q,\cdot)]-g_{\partial\mathcal{S}_0,j}(\varepsilon,q,\cdot)\Big).\end{aligned}$$

Now taking into account the definitions of C_j^1 and $\mathsf{C}_j^2(q)$ and the boundary conditions we get that (5.98b) holds true.

Proof of (5.98c). Finally it remains to check (5.98c). First we obtain, on $\partial\Omega$, that

$$SL[p^{\varepsilon}_{\partial\Omega,j}(q,\cdot)] = \varepsilon^2\, \overline{\varphi}_{\cancel{\partial\mathcal{S}},j}(q,\cdot) + \varepsilon^3\, SL[\mathsf{p}_{\partial\Omega,r,j}(\varepsilon,q,\cdot)], \tag{5.116}$$

and

$$\begin{aligned} K_{\partial\Omega}(\varepsilon,q)[p^{\varepsilon}_{\partial\mathcal{S}_0,j}(q,\cdot)](x) = &-\varepsilon^2\, R(\vartheta)^t\, DG(x-h)\cdot T^1(\mathsf{p}_{\cancel{\partial\Omega},j}) \\ &+ \varepsilon^3\left(K_{\partial\Omega}(\varepsilon,q)[p^{\varepsilon}_{\partial\mathcal{S}_0,r}(\varepsilon,q,\cdot)](x) - g_{\partial\Omega}(\varepsilon,q,x)\right). \end{aligned} \tag{5.117}$$

Gathering (5.117) and (5.116), the right hand side of the equation (5.98c) now reads, for $x\in\partial\Omega$,

$$\begin{aligned} SL[p^{\varepsilon}_{\partial\Omega,j}(q,\cdot)](x) + K_{\partial\Omega}(\varepsilon,q)[p^{\varepsilon}_{\partial\mathcal{S}_0,j}(q,\cdot)](x) = &\ \varepsilon^2\left(\overline{\varphi}_{\cancel{\partial\mathcal{S}}}(q,x) - R(\vartheta)^t\, DG(x-h)\cdot T^1(\mathsf{p}_{\cancel{\partial\Omega},j})\right) \\ &+ \varepsilon^3\left(SL[\mathsf{p}_{\partial\Omega,r,j}(\varepsilon,q,\cdot)] + K_{\partial\Omega}(\varepsilon,q)[p^{\varepsilon}_{\partial\mathcal{S}_0,r,j}(\varepsilon,q,\cdot)](x) - g_{\partial\Omega,j}(\varepsilon,q,x)\right). \end{aligned} \tag{5.118}$$

Taking now the boundary conditions into account, we deduce from the equation (5.118) that (5.98c) holds true. □

Third Step. Existence and estimate of the reminders. We now focus on equation (5.111). We apply Lemma 15 with $\mathfrak{g}_j(\varepsilon,q,\cdot)$ given by (5.109) instead of $\mathfrak{g}(\varepsilon,q,\cdot)$. Thanks to (5.110) the assumption of Lemma 15 is verified and we therefore obtain the existence of $\varepsilon_0\in(0,1)$ and of $\mathfrak{p}_{r,j}\in L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0};F_0)$ such that for any $(\varepsilon,q)\in\mathfrak{Q}_{\delta,\varepsilon_0}$, $\mathfrak{p}_{r,j}(\varepsilon,q,\cdot)$ solves (5.111).

Fourth Step. Conclusion. Combining with Lemma 20, (5.103), (5.104) and the jump formulas:

$$\begin{aligned} \text{on } \partial\mathcal{S}_0, &\quad \mathsf{p}_{\cancel{\partial\Omega},j}(q,\cdot) = \frac{\partial\overline{\varphi}_{\cancel{\partial\Omega},j}}{\partial n} - \frac{\partial\kappa_j^1}{\partial n}(0,\cdot), \\ \text{on } \partial\Omega, &\quad \mathsf{p}_{\cancel{\partial\mathcal{S}},j}(q,\cdot) = \frac{\partial\overline{\varphi}_{\cancel{\partial\mathcal{S}},j}}{\partial n}(q,\cdot), \end{aligned}$$

we get

$$\begin{aligned} \text{on } \partial\mathcal{S}_0, &\quad \frac{\partial\overline{\varphi}^{\varepsilon}_j}{\partial n}(q,\varepsilon R(\vartheta)\cdot+h) = \varepsilon^{\delta_{1,j}}\left(\frac{\partial\overline{\varphi}_{\cancel{\partial\Omega},j}}{\partial n} + \varepsilon^2\, \mathsf{p}_{\partial\mathcal{S}_0,r,j}(\varepsilon,q,\cdot)\right), \\ \text{on } \partial\Omega, &\quad \frac{\partial\overline{\varphi}^{\varepsilon}_j}{\partial n}(q,\cdot) = \varepsilon^{\delta_{1,j}}\left(\varepsilon^2\frac{\partial\overline{\varphi}_{\cancel{\partial\mathcal{S}_0},j}}{\partial n}(q,\cdot) + \varepsilon^3\, \mathsf{p}_{\partial\Omega,r,j}(\varepsilon,q,\cdot)\right), \end{aligned}$$

with $\mathsf{p}_{\partial\mathcal{S}_0,r,j}\in L^\infty\big(\mathfrak{Q}_{\delta,\varepsilon_0};L^2(\partial\mathcal{S}_0;\mathbb{R})\big)$ and $\mathsf{p}_{\partial\Omega,r,j}\in L^\infty\big(\mathfrak{Q}_{\delta,\varepsilon_0};L^2(\partial\Omega;\mathbb{R})\big)$.

It is then sufficient to set, for $(\varepsilon,q)\in\mathfrak{Q}_{\delta,\varepsilon_0}$,

$$\boldsymbol{p}_{\partial\mathcal{S}_0,r}(\varepsilon,q,\cdot) := \big(\mathsf{p}_{\partial\mathcal{S}_0,r,j}(\varepsilon,q,\cdot)\big)_{j=1,2,3}, \quad \text{and} \quad \boldsymbol{p}_{\partial\Omega,r}(\varepsilon,q,\cdot) := \Big(\frac{\partial\overline{\varphi}_{\cancel{\partial\mathcal{S}_0},j}}{\partial n}(q,\cdot) + \varepsilon\mathsf{p}_{\partial\Omega,r,j}(\varepsilon,q,\cdot)\Big)_{j=1,2,3},$$

so that $\boldsymbol{p}_{\partial\mathcal{S}_0,r}\in L^\infty\big(\mathfrak{Q}_{\delta,\varepsilon_0};L^2(\partial\mathcal{S}_0;\mathbb{R}^3)\big)$ and $\boldsymbol{p}_{\partial\Omega,r}(\varepsilon,q,\cdot)\in L^\infty\big(\mathfrak{Q}_{\delta,\varepsilon_0};L^2(\partial\Omega;\mathbb{R}^3)\big)$, to conclude the proof of Proposition 9. □

6. Asymptotic expansion of the added inertia and the Christoffel symbols

In this section, we use the asymptotic developments of Section 5 to deduce expansions for the added inertia matrix and for the Christoffel symbols.

6.1. **Asymptotic expansion of the added inertia $M_a^\varepsilon(q)$: Proof of Proposition 4.** The matrix $M_a^\varepsilon(q)$ is the counterpart for the body of size ε of the added mass $M_a(q)$ defined in (1.9b). It is defined for $(\varepsilon, q) \in \mathfrak{Q}$ by

$$M_a^\varepsilon(q) := \int_{\partial \mathcal{S}^\varepsilon(q)} \boldsymbol{\varphi}^\varepsilon(q,\cdot) \otimes \frac{\partial \boldsymbol{\varphi}^\varepsilon}{\partial n}(q,\cdot)\mathrm{d}s = \int_{\partial \mathcal{S}^\varepsilon(q)} \boldsymbol{\varphi}^\varepsilon(q,\cdot) \otimes \boldsymbol{K}^\varepsilon(q,\cdot)\mathrm{d}s. \tag{6.1}$$

The function $\boldsymbol{\varphi}^\varepsilon$ mentioned above is defined in (5.28), (5.29). Using a change of variable, observing that, for $(\varepsilon, q) \in \mathfrak{Q}$, on $\partial\mathcal{S}_0$,

$$\boldsymbol{K}^\varepsilon(q, \varepsilon R(\vartheta)\cdot + h) = I_\varepsilon \mathcal{R}(\vartheta)\boldsymbol{K}(0,\cdot), \tag{6.2}$$

we obtain:

$$M_a^\varepsilon(q) = \varepsilon \int_{\partial\mathcal{S}_0} \boldsymbol{\varphi}^\varepsilon(q, \varepsilon R(\vartheta)\cdot + h) \otimes I_\varepsilon \mathcal{R}(\vartheta)\boldsymbol{K}^1(0,\cdot)\mathrm{d}s. \tag{6.3}$$

We now apply Proposition 8, (i) to get

$$M_a^\varepsilon(q) = \varepsilon^2 I_\varepsilon \Big(\int_{\partial\mathcal{S}_0} \mathcal{R}(\vartheta)\Big(\boldsymbol{\varphi}_{\cancel{\partial\Omega}} + \check{\boldsymbol{c}}(\varepsilon,q) + \varepsilon^2 \boldsymbol{\varphi}_r(\varepsilon,q,\cdot)\Big) \otimes \mathcal{R}(\vartheta)\boldsymbol{K}^1(0,\cdot)\mathrm{d}s\Big) I_\varepsilon \tag{6.4}$$

$$= \varepsilon^2 I_\varepsilon \Big(M_{a,\cancel{\partial\Omega},\vartheta} + \varepsilon^2 \mathcal{R}(\vartheta) \int_{\partial\mathcal{S}_0} \boldsymbol{\varphi}_r(\varepsilon,q,\cdot) \otimes \boldsymbol{K}^1(0,\cdot)\mathrm{d}s\, \mathcal{R}(\vartheta)^t \Big) I_\varepsilon, \tag{6.5}$$

since

$$\int_{\partial\mathcal{S}_0} \check{\boldsymbol{c}}(\varepsilon,q) \otimes \boldsymbol{K}^1(0,\cdot)\mathrm{d}s = \check{\boldsymbol{c}}(\varepsilon,q) \otimes \int_{\partial\mathcal{S}_0} \boldsymbol{K}^1(0,\cdot)\mathrm{d}s = 0,$$

and

$$M_{a,\cancel{\partial\Omega},\vartheta} = \mathcal{R}(\vartheta) \int_{\partial\mathcal{S}_0} \boldsymbol{\varphi}_{\cancel{\partial\Omega}} \otimes \boldsymbol{K}^1(0,\cdot)\mathrm{d}s\, \mathcal{R}(\vartheta)^t,$$

thanks to (2.8b), (2.10) and (2.11). Above $\varepsilon_0 \in (0,1)$ and $\boldsymbol{\varphi}_r \in L^\infty\big(\mathfrak{Q}_{\delta,\varepsilon_0}; L^2(\partial\mathcal{S}_0;\mathbb{R}^3)\big)$.

Then we set

$$M_r(\varepsilon,q) := \mathcal{R}(\vartheta) \int_{\partial\mathcal{S}_0} \boldsymbol{\varphi}_r(\varepsilon,q,\cdot) \otimes \boldsymbol{K}^1(0,\cdot)\mathrm{d}s\, \mathcal{R}(\vartheta)^t,$$

and we observe that M_r is in $L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0};\mathbb{R})$ and depends only on $\mathcal{S}_0$ and Ω, which concludes the proof of Proposition 4.

6.2. **Expansion of $\Gamma_{\mathcal{S}}$.** In this subsection we consider the Christoffel symbols $\Gamma_{\mathcal{S}}^\varepsilon$ given for $(\varepsilon,q) \in \mathfrak{Q}$ and $p \in \mathbb{R}^3$, by

$$\langle \Gamma_{\mathcal{S}}^\varepsilon(q), p, p\rangle := -\begin{pmatrix} 0 \\ P_a^\varepsilon \end{pmatrix} \times p - \omega M_a^\varepsilon(q) \begin{pmatrix} 0 \\ \ell^\perp \end{pmatrix} \in \mathbb{R}^3, \tag{6.6}$$

where P_a^ε denotes the two last lines of $M_a^\varepsilon(q)p$. The formula (6.6) is the counterpart for a body of size ε of the Christoffel symbols given by (3.1) when $\varepsilon = 1$.

The next result proves that the leading term of $\Gamma_{\mathcal{S}}^\varepsilon$ is given, up to an appropriate scaling, by the Christoffel symbols $\langle \Gamma_{\cancel{\partial\Omega},\vartheta}, p, p\rangle$ of the solid as if it was immersed in a fluid filling the plane. We recall that $\langle \Gamma_{\cancel{\partial\Omega},\vartheta}, p, p\rangle$ is defined in (2.13). Precisely, we have the following result.

Proposition 10. *Let $\delta > 0$. There exists $\varepsilon_0 \in (0,1)$ and $\Gamma_{\mathcal{S},r} \in L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0}; \mathcal{BL}(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R}^3))$ depending on $\mathcal{S}_0$, γ and Ω, such that for any (ε,q) in $\mathfrak{Q}_{\delta,\varepsilon_0}$, with $q = (\vartheta, h)$, for any $p = (\omega, \ell) \in \mathbb{R}^3$,*

$$\langle \Gamma_{\mathcal{S}}^\varepsilon(q), p, p\rangle = \varepsilon I_\varepsilon \big(\langle \Gamma_{\cancel{\partial\Omega},\vartheta}, \hat{p}, \hat{p}\rangle + \varepsilon^2 \langle \Gamma_{\mathcal{S},r}(\varepsilon,q), \hat{p}, \hat{p}\rangle \big), \tag{6.7}$$

where $\hat{p} = (\hat{\omega}, \ell) = (\varepsilon\omega, \ell)$.

Proposition 10 follows from Proposition 4 by straightforward computations.

6.3. **Expansion of $\Gamma_{\partial\Omega}$.** In this subsection we study the Christoffel symbols $\Gamma^\varepsilon_\Omega$ given for $(\varepsilon, q) \in \mathfrak{Q}$ and $p = (p_1, p_2, p_3) \in \mathbb{R}^3$, by

$$\langle \Gamma^\varepsilon_{\partial\Omega}(q), p, p\rangle := \left(\sum_{1\leqslant k,l\leqslant 3} (\Gamma^\varepsilon_{\partial\Omega})^j_{k,l}(q) p_k p_l \right)_{1\leqslant j\leqslant 3} \in \mathbb{R}^3, \tag{6.8}$$

where for every $j, k, l \in \{1, 2, 3\}$, we set

$$(\Gamma^\varepsilon_{\partial\Omega})^j_{k,l}(q) := \frac{1}{2}[\Lambda^{\varepsilon,l}_{kj}(q) + \Lambda^{\varepsilon,k}_{jl}(q) - \Lambda^{\varepsilon,j}_{kl}(q)],$$

with

$$\Lambda^{\varepsilon,l}_{kj}(q) := \int_{\partial\Omega} \left(\frac{\partial\varphi^\varepsilon_j}{\partial\tau} \frac{\partial\varphi^\varepsilon_k}{\partial\tau} K_l \right)(q, \cdot)\, \mathrm{d}s.$$

Proposition 11. *Let $\delta > 0$. There exists $\varepsilon_0 \in (0, 1)$ and $\Gamma_{\partial\Omega,r} \in L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0}; \mathcal{BL}(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R}^3))$ depending on $\mathcal{S}_0$, γ and Ω, such that for any (ε, q) in $\mathfrak{Q}_{\delta,\varepsilon_0}$, with $q := (\vartheta, h)$, for any $p := (\omega, \ell) \in \mathbb{R}^3$,*

$$\langle \Gamma^\varepsilon_{\partial\Omega}(q), p, p\rangle = \varepsilon^3 I_\varepsilon \langle \Gamma_{\partial\Omega,r}(\varepsilon, q), \hat{p}, \hat{p}\rangle. \tag{6.9}$$

Proof of Proposition 11. Proposition 11 follows from Proposition 8, (iii). Indeed, (6.8) can be rewritten as:

$$\langle \Gamma^\varepsilon_{\partial\Omega}(q), p, p\rangle = \int_{\partial\Omega} \left[\frac{\partial\boldsymbol{\varphi}^\varepsilon}{\partial\tau} (\mathbf{K}^\varepsilon \cdot p) \left(\frac{\partial\boldsymbol{\varphi}^\varepsilon}{\partial\tau} \cdot p \right) - \frac{1}{2} \mathbf{K}^\varepsilon \left(\frac{\partial\boldsymbol{\varphi}^\varepsilon}{\partial\tau} \cdot p \right)^2 \right] (q, \cdot)\, \mathrm{d}s.$$

Observe that $\mathbf{K}^\varepsilon$ is actually independent of ε on $\partial\Omega$. Let us denote $\hat{\mathbf{K}}^\varepsilon := \varepsilon I_\varepsilon^{-1} \mathbf{K}^\varepsilon$. According to (5.32), we obtain:

$$\langle \Gamma^\varepsilon_{\partial\Omega}(q), p, p\rangle = \varepsilon^3 I_\varepsilon \int_{\partial\Omega} \left[\mathrm{p}_{\partial\Omega,r} (\hat{\mathbf{K}}^\varepsilon \cdot \hat{p}) \left(\mathrm{p}_{\partial\Omega,r} \cdot \hat{p} \right) - \frac{1}{2} \hat{\mathbf{K}}^\varepsilon \left(\mathrm{p}_{\partial\Omega,r} \cdot \hat{p} \right)^2 \right] (q, \cdot)\, \mathrm{d}s,$$

which gives the expected result. □

7. Asymptotic expansion of the total force H: Proof of Proposition 5

This section is devoted to the proof of Proposition 5. We start with recalling a technical result borrowed from [14, Article 134a. (3) and (7)], cf. Lemma 22 below. We will go on with some technical results regarding some inertia matrices and the corrector velocity. Then we will make use of the expansions of the previous sections and of Lemma 22 to expand the different contributions coming from E and B. Finally we will combine these expansions, together with those of the Christoffel symbols obtained above, in order to conclude the proof of Proposition 5.

7.1. **Lamb's lemma.** The following lemma seems to originate from Lamb's work. We recall that ξ_j and K_j, for $j = 1, 2, 3$, were defined in (1.5) and (1.6) respectively.

Lemma 22. *For any pair of vector fields u, v in $C^\infty(\overline{\mathbb{R}^2 \setminus \mathcal{S}_0}; \mathbb{R}^2)$ satisfying*

- $\operatorname{div} u = \operatorname{div} v = \operatorname{curl} u = \operatorname{curl} v = 0$,
- *$u(x) = O(1/|x|)$ and $v(x) = O(1/|x|)$ as $|x| \to +\infty$,*

one has, for any $j = 1, 2, 3$,

$$\int_{\partial\mathcal{S}_0} (u \cdot v) K_j(0, \cdot) \mathrm{d}s = \int_{\partial\mathcal{S}_0} \xi_j(0, \cdot) \cdot \Big((u \cdot n)v + (v \cdot n)u \Big) \mathrm{d}s. \tag{7.1}$$

Proof of Lemma 22. Let us start with the case where $j = 2$ or 3. Then

$$\int_{\partial\mathcal{S}_0} (u\cdot v)K_j(0,\cdot)\mathrm{d}s = \int_{\partial\mathcal{S}_0} \big((u\cdot v)\xi_j(0,\cdot)\big)\cdot n\mathrm{d}s = \int_{\mathbb{R}^2\setminus\mathcal{S}_0} \mathrm{div}\,\big((u\cdot v)\xi_j(0,\cdot)\big)\,\mathrm{d}x, \tag{7.2}$$

by using that $u(x) = O(1/|x|)$ and $v(x) = O(1/|x|)$ when $|x| \to +\infty$. Therefore

$$\int_{\partial\mathcal{S}_0} (u\cdot v)K_j(0,\cdot)\mathrm{d}s = \int_{\mathbb{R}^2\setminus\mathcal{S}_0} \xi_j(0,\cdot)\cdot\nabla(u\cdot v)\,\mathrm{d}x = \int_{\mathbb{R}^2\setminus\mathcal{S}_0} \xi_j(0,\cdot)\cdot(u\cdot\nabla v + v\cdot\nabla u)\,\mathrm{d}x, \tag{7.3}$$

using that $\mathrm{curl}\,u = \mathrm{curl}\,v = 0$. Now, integrating by parts, using that $\mathrm{div}\,u = \mathrm{div}\,v = 0$ and once again that $u(x) = O(1/|x|)$ and $v(x) = O(1/|x|)$ as $|x| \to +\infty$, we obtain (7.1) when $j = 2$ or 3.

We now tackle the case where $j = 1$. We follow the same lines as above, with two precisions. First we observe that there is no contribution at infinity in (7.2) and (7.3) when $j = 1$ as well. Indeed ξ_1 and the normal to a centered circle are orthogonal. Moreover there is no additional distributed term coming from the integration by parts in (7.3) when $j = 1$ since

$$\int_{\mathbb{R}^2\setminus\mathcal{S}_0} v\cdot(u\cdot\nabla_x\xi_j(0,\cdot)) + u\cdot(v\cdot\nabla_x\xi_j(0,\cdot))\,\mathrm{d}x = \int_{\mathbb{R}^2\setminus\mathcal{S}_0} (v\cdot u^\perp + u\cdot v^\perp)\,\mathrm{d}x = 0.$$

□

7.2. **Some useful inertia matrices.** In the sequel it will be useful to consider some functions of the entries of the matrix $M_{a,\cancel{\partial\Omega}}$ defined in (2.10). We decompose $M_{a,\cancel{\partial\Omega}}$ into

$$M_{a,\cancel{\partial\Omega}} =: \begin{pmatrix} m^\# & \mu^t \\ \mu & M^\flat \end{pmatrix}, \tag{7.4}$$

where $M^\flat$ is a symmetric 2×2 matrix. We also define the real traceless symmetric 2×2 matrix $M^\dagger$ defined by

$$M^\dagger = \big(M^\dagger_{i,j}\big)_{1\leqslant i,j\leqslant 2} := \frac{1}{2}\Big(M^\flat(\perp) + \big(M^\flat(\perp)\big)^t\Big) = \frac{1}{2}\Big(M^\flat(\perp) - (\perp)M^\flat\Big), \tag{7.5}$$

where $(\perp)$ denotes the 2×2 matrix

$$(\perp) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{7.6}$$

The matrix $M^\dagger$ depends only on $\mathcal{S}_0$. Its coefficients can be described as follows: using the definition of the matrix $M_{a,\cancel{\partial\Omega}}$ in (2.10), we have

$$M^\dagger_{1,1} = -M^\dagger_{2,2} = \int_{\partial\mathcal{S}_0} \frac{\partial\varphi_{\cancel{\partial\Omega},3}}{\partial n}\varphi_{\cancel{\partial\Omega},2}\mathrm{d}s, \quad \text{and} \tag{7.7a}$$

$$M^\dagger_{1,2} = M^\dagger_{2,1} = \frac{1}{2}\int_{\partial\mathcal{S}_0} \Big(\frac{\partial\varphi_{\cancel{\partial\Omega},3}}{\partial n}\varphi_{\cancel{\partial\Omega},3} - \frac{\partial\varphi_{\cancel{\partial\Omega},2}}{\partial n}\varphi_{\cancel{\partial\Omega},2}\Big)\mathrm{d}s, \tag{7.7b}$$

where the functions $\varphi_{\cancel{\partial\Omega},j}$, for $j = 1, 2, 3$, are the Kirchhoff's potentials in $\mathbb{R}^2\setminus\mathcal{S}_0$ defined in (2.8).

We also consider

$$M^\flat_\vartheta := \mathcal{R}(\vartheta)M^\flat\mathcal{R}(\vartheta)^t, \quad \mu_\vartheta := R(\vartheta)\mu, \quad M^\dagger_\vartheta := R(\vartheta)M^\dagger R(\vartheta)^t. \tag{7.8}$$

The matrix $M^\dagger_\vartheta$ enjoys the following properties.

Lemma 23. *For any $\vartheta \in \mathbb{R}$, for any $X \in \mathbb{R}^2$,*

$$(M_\vartheta^\flat X)^\perp \cdot X = X^\perp \cdot M_\vartheta^\dagger X^\perp, \tag{7.9}$$

$$(M_\vartheta^\flat X)^\perp - M_\vartheta^\flat X^\perp = -2 M_\vartheta^\dagger X, \tag{7.10}$$

$$(\perp) M_\vartheta^\dagger (\perp) = M_\vartheta^\dagger. \tag{7.11}$$

The proof of Lemma 23 is elementary and left to the reader.

In Section 7.5 we will need to compute $\langle \Gamma_{\cancel{\partial\Omega},\vartheta}, p, p\rangle$ in terms of $M^\dagger$.

Lemma 24. *For any $\vartheta \in \mathbb{R}$, for any $p = (\omega, \ell) \in \mathbb{R}^3$,*

$$\langle \Gamma_{\cancel{\partial\Omega},\vartheta}, p, p\rangle = \begin{pmatrix} -\ell^\perp \cdot M_\vartheta^\dagger \, \ell^\perp \\ \omega^2 \mu_\vartheta^\perp - 2\omega M_\vartheta^\dagger \, \ell \end{pmatrix}. \tag{7.12}$$

Proof of Lemma 24. Using the definition of $M^1_{a,\cancel{\partial\Omega},\vartheta}$ in (2.11) and the decomposition of $M_{a,\cancel{\partial\Omega}}$ in (7.4) we get

$$M_{a,\cancel{\partial\Omega},\vartheta} = \begin{pmatrix} m^\# & \mu_\vartheta^t \\ \mu_\vartheta & M_\vartheta^\flat \end{pmatrix}, \tag{7.13}$$

with $M_\vartheta^\flat$ and μ_ϑ as in (7.8). As a consequence

$$P_{a,\cancel{\partial\Omega},\vartheta} = \omega \mu_\vartheta + M_\vartheta^\flat \ell. \tag{7.14}$$

In particular we infer from (2.20), (2.13), (7.13) and (7.14) that

$$\begin{aligned} \langle \Gamma_{\cancel{\partial\Omega},\vartheta}, p, p\rangle &= -\begin{pmatrix} 0 \\ \omega\mu_\vartheta + M_\vartheta^\flat \ell \end{pmatrix} \times \begin{pmatrix} \omega \\ \ell \end{pmatrix} - \omega \begin{pmatrix} \mu_\vartheta \cdot \ell^\perp \\ M_\vartheta^\flat \ell^\perp \end{pmatrix} \\ &= -\begin{pmatrix} \omega \mu_\vartheta^\perp \cdot \ell + (M_\vartheta^\flat \ell)^\perp \cdot \ell \\ -\omega^2 \mu_\vartheta^\perp - \omega (M_\vartheta^\flat \ell)^\perp \end{pmatrix} - \omega \begin{pmatrix} \mu_\vartheta \cdot \ell^\perp \\ M_\vartheta^\flat \ell^\perp \end{pmatrix} \\ &= \begin{pmatrix} -(M_\vartheta^\flat \ell)^\perp \cdot \ell \\ \omega^2 \mu_\vartheta^\perp + \omega\big((M_\vartheta^\flat \ell)^\perp - M_\vartheta^\flat \ell^\perp\big) \end{pmatrix}. \end{aligned}$$

It remains to recast this expression thanks to the matrix $M^\dagger$ defined in (7.5). This is done thanks to Lemma 23. □

Let us introduce now the matrix

$$\overline{M} := \int_{\partial \mathcal{S}_0} \begin{pmatrix} \overline{\varphi}_{\cancel{\partial\Omega},2} \\ \overline{\varphi}_{\cancel{\partial\Omega},3} \end{pmatrix} \otimes \begin{pmatrix} \frac{\partial \overline{\varphi}_{\cancel{\partial\Omega},3}}{\partial n} \\ -\frac{\partial \overline{\varphi}_{\cancel{\partial\Omega},2}}{\partial n} \end{pmatrix} \mathrm{d}s, \tag{7.15}$$

where the functions $\overline{\varphi}_{\cancel{\partial\Omega},j}$, for $j = 1, 2, 3$, defined in (5.86), are harmonic conjugates to the functions $\varphi_{\cancel{\partial\Omega},j}$. The matrix $\overline{M}$ will intervene through the following result regarding the stream function $\psi^0_{\cancel{\partial\Omega}}(q, \cdot)$ defined in (5.17).

Lemma 25. *For any $q := (\vartheta, h)$ in $\mathbb{R} \times \Omega$,*

$$\int_{\partial \mathcal{S}_0} \frac{\partial \psi^0_{\cancel{\partial\Omega}}}{\partial n}(q, x) x^\perp \, \mathrm{d}s(x) = \overline{M} R(\vartheta)^t \, u_\Omega(h)^\perp. \tag{7.16}$$

Proof of Lemma 25. First it follows from (5.86b) that, on $\partial\mathcal{S}_0$,

$$x^\perp = \begin{pmatrix} \overline{\varphi}_{\cancel{\partial\Omega},2} - c_{\cancel{\partial\Omega},2} \\ \overline{\varphi}_{\cancel{\partial\Omega},3} - c_{\cancel{\partial\Omega},3} \end{pmatrix}. \tag{7.17}$$

We now express the stream function $\psi^0_{\cancel{\partial\Omega}}(q,\cdot)$ thanks to the functions $\overline{\varphi}_{\cancel{\partial\Omega},3}$ and $\overline{\varphi}_{\cancel{\partial\Omega},2}$.

Lemma 26. *For any $q := (\vartheta, h)$ in $\mathbb{R}\times\Omega$, on $\overline{\mathbb{R}^2 \setminus \mathcal{S}_0}$,*

$$\psi^0_{\cancel{\partial\Omega}}(q,\cdot) = R(\vartheta)^t\, u_\Omega(h)^\perp \cdot \begin{pmatrix} \overline{\varphi}_{\cancel{\partial\Omega},3} \\ -\overline{\varphi}_{\cancel{\partial\Omega},2} \end{pmatrix} \tag{7.18}$$

Proof of Lemma 26. Let $q := (\vartheta, h)$ in $\mathbb{R}\times\Omega$. On $\partial\mathcal{S}_0$, it follows from (5.17b) and (5.86) that the equality holds true up to a constant. Therefore according to Corollary 6 there exists $c \in \mathbb{R}$ such that, on $\overline{\mathbb{R}^2 \setminus \mathcal{S}_0}$,

$$\psi^0_{\cancel{\partial\Omega}}(q,\cdot) = R(\vartheta)^t\, u_\Omega(h)^\perp \cdot \begin{pmatrix} \overline{\varphi}_{\cancel{\partial\Omega},3} \\ -\overline{\varphi}_{\cancel{\partial\Omega},2} \end{pmatrix} + c\, \psi^{-1}_{\cancel{\partial\Omega}}.$$

Then using (2.16e), (5.17c) and (5.86d) we obtain $c = 0$, which proves the result. □

As a consequence, we get, on $\partial\mathcal{S}_0$,

$$\frac{\partial \psi^0_{\cancel{\partial\Omega}}}{\partial n}(q,\cdot) = R(\vartheta)^t\, u_\Omega(h)^\perp \cdot \begin{pmatrix} \frac{\partial \overline{\varphi}_{\cancel{\partial\Omega},3}}{\partial n} \\ -\frac{\partial \overline{\varphi}_{\cancel{\partial\Omega},2}}{\partial n} \end{pmatrix}. \tag{7.19}$$

Plugging (7.17) and (7.19) into the left hand side of (7.16) and using (5.17c) establishes Lemma 25. □

We will also use the following identities regarding the analogous moments of the functions $\frac{\partial \overline{\varphi}_{\cancel{\partial\Omega},3}}{\partial \tau}$ and $\frac{\partial \overline{\varphi}_{\cancel{\partial\Omega},2}}{\partial \tau}$.

Lemma 27. *The following identities hold true:*

$$\int_{\partial\mathcal{S}_0} \frac{\partial \overline{\varphi}_{\cancel{\partial\Omega},3}}{\partial \tau}(x) x^\perp \, \mathrm{d}s(x) = -\overline{M}\xi_2(0,\cdot), \tag{7.20}$$

$$\int_{\partial\mathcal{S}_0} \frac{\partial \overline{\varphi}_{\cancel{\partial\Omega},2}}{\partial \tau}(x) x^\perp \, \mathrm{d}s(x) = \overline{M}\xi_3(0,\cdot). \tag{7.21}$$

Proof of Lemma 27. Let us focus on the proof of (7.20), the proof of (7.21) being similar. Using an integration by parts we see that

$$\int_{\partial\mathcal{S}_0} \frac{\overline{\varphi}_{\cancel{\partial\Omega},3}}{\partial \tau}(x) x^\perp \, \mathrm{d}s(x) = -\int_{\partial\mathcal{S}_0} \varphi_{\cancel{\partial\Omega},3}\, n \, \mathrm{d}s = -\overline{M}\xi_2(0,\cdot).$$

□

Let us now connect the matrices $M^\dagger$ and $\overline{M}$.

Lemma 28. *The following identity holds true:*

$$M^\dagger = \frac{1}{2}(\overline{M} + \overline{M}^t). \tag{7.22}$$

Proof of Lemma 28. Using some integrations by parts and (5.87), we get, for any $i, j = 2, 3$,

$$\begin{aligned}\int_{\partial\mathcal{S}_0} \frac{\partial \overline{\varphi}_{\partial\Omega,i}}{\partial n} \overline{\varphi}_{\partial\Omega,j} \mathrm{d}s &= \int_{\mathbb{R}^2 \setminus \mathcal{S}_0} \nabla \overline{\varphi}_{\partial\Omega,i} \cdot \nabla \overline{\varphi}_{\partial\Omega,j} \mathrm{d}x \\ &= \int_{\mathbb{R}^2 \setminus \mathcal{S}_0} \nabla \varphi_{\partial\Omega,i} \cdot \nabla \varphi_{\partial\Omega,j} \mathrm{d}x \\ &= \int_{\partial\mathcal{S}_0} \frac{\partial \varphi_{\partial\Omega,i}}{\partial n} \varphi_{\partial\Omega,j} \mathrm{d}x.\end{aligned}$$

Combining this with (7.7) and recalling the definition of $\overline{M}$ in (7.15) yield (7.22). □

7.3. **Expansion of E^ε.**

We now consider the expansion of E^ε which is given, for $(\varepsilon, q) \in \mathfrak{Q}$, by

$$E^\varepsilon(q) := -\frac{1}{2} \int_{\partial\mathcal{S}^\varepsilon(q)} \left| \frac{\partial \psi^\varepsilon}{\partial n}(q, \cdot) \right|^2 \boldsymbol{K}^\varepsilon(q, \cdot) \mathrm{d}s.$$

This formula is the counterpart of (1.15b) for a body of size ε. We recall that the function $\psi^\varepsilon(q, \cdot)$ is defined in (5.14) and the vector field $\boldsymbol{K}^\varepsilon(q, \cdot)$ in (5.27)-(5.29).

The following expansion will use the vector fields $\mathsf{E}^1_b(q)$ and $\mathsf{E}^1_c(q)$ defined in (4.5) and (4.7) and the following ones:

$$\mathsf{E}^0(q) := -\begin{pmatrix} u_\Omega(h) \cdot \zeta_\vartheta \\ u_\Omega(h)^\perp \end{pmatrix} \quad \text{and} \quad \mathsf{E}^1_a(q) := \begin{pmatrix} u_\Omega(h)^\perp M^\dagger_\vartheta \, u_\Omega(h)^\perp \\ 0 \\ 0 \end{pmatrix}. \tag{7.23}$$

We recall that u_Ω, ζ_ϑ and $M^\dagger_\vartheta$ were defined in (1.26), (2.15)-(2.17) and (7.5)-(7.8), respectively.

The goal of this subsection is to establish the following result.

Proposition 12. *Let $\delta > 0$. There exists $\varepsilon_0 \in (0, 1)$ and a function $E_r \in L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0}; \mathbb{R}^3)$ depending on $\mathcal{S}_0$ and Ω, such that for any (ε, q) in $\mathfrak{Q}_{\delta,\varepsilon_0}$,*

$$E^\varepsilon(q) = I_\varepsilon \Big(\mathsf{E}^0(q) + \varepsilon \mathsf{E}^1(q) + \varepsilon^2 E_r(\varepsilon, q) \Big), \tag{7.24}$$

where

$$\mathsf{E}^1(q) := \mathsf{E}^1_a(q) + \mathsf{E}^1_b(q) + \mathsf{E}^1_c(q). \tag{7.25}$$

Proof of Proposition 12. Let $\delta > 0$. We proceed in three steps: first we use a change of variable in order to recast $E^\varepsilon(q)$ as an integral on the fixed boundary $\partial\mathcal{S}_0$. Then we plug the expansion of ψ^ε into this integral. Finally we use several times Lamb's lemma in order to compute the terms of the resulting expansion.

Thanks to a change of variable, using (6.2), we get

$$E^\varepsilon(q) = -\frac{\varepsilon}{2} I_\varepsilon \mathcal{R}(\vartheta) \int_{\partial\mathcal{S}_0} \left| \frac{\partial \psi^\varepsilon}{\partial n}(q, \varepsilon R(\vartheta) \cdot + h) \right|^2 \boldsymbol{K}(0, \cdot) \mathrm{d}s,$$

where $\boldsymbol{K}(q, \cdot)$ is the vector field defined in (1.8). Using (5.26) we get that there exists $\varepsilon_0 \in (0, 1)$ such that for any (ε, q) in $\mathfrak{Q}_{\delta,\varepsilon_0}$,

$$E^\varepsilon(q) = I_\varepsilon \mathcal{R}(\vartheta) \Big(\frac{1}{\varepsilon} \underline{\mathsf{E}}^{-1} + \underline{\mathsf{E}}^0(q) + \varepsilon \underline{\mathsf{E}}^1(q) + \varepsilon^2 \underline{E}_r(\varepsilon, q) \Big), \tag{7.26}$$

with

$$\underline{\mathsf{E}}^{-1} := -\frac{1}{2}\int_{\partial\mathcal{S}_0}\left|\frac{\partial\psi^{-1}_{\cancel{\partial\Omega}}}{\partial n}\right|^2 \boldsymbol{K}(0,\cdot)\mathrm{d}s,$$

$$\underline{\mathsf{E}}^{0}(q) := -\int_{\partial\mathcal{S}_0}\frac{\partial\psi^{-1}_{\cancel{\partial\Omega}}}{\partial n}\Big(\frac{\partial\psi^{0}_{\cancel{\partial\Omega}}}{\partial n}(q,\cdot) - R(\vartheta)^t\, u_\Omega(h)\cdot\tau\Big)\boldsymbol{K}(0,\cdot)\mathrm{d}s, \tag{7.27}$$

$$\underline{\mathsf{E}}^{1}(q) := \underline{\mathsf{E}}^{1}_a(q) + \underline{\mathsf{E}}^{1}_b(q), \tag{7.28}$$

where

$$\underline{\mathsf{E}}^{1}_a(q) := -\frac{1}{2}\int_{\partial\mathcal{S}_0}\left|\frac{\partial\psi^{0}_{\cancel{\partial\Omega}}}{\partial n}(q,\cdot) - R(\vartheta)^t\, u_\Omega(h)\cdot\tau\right|^2 \boldsymbol{K}(0,\cdot)\mathrm{d}s, \tag{7.29}$$

$$\underline{\mathsf{E}}^{1}_b(q) := -\int_{\partial\mathcal{S}_0}\frac{\partial\psi^{-1}_{\cancel{\partial\Omega}}}{\partial n}\Big(\frac{\partial\psi^{1}_{\cancel{\partial\Omega}}}{\partial n}(q,\cdot) - \frac{\partial P^1}{\partial n}(q,\cdot)\Big)\boldsymbol{K}(0,\cdot)\mathrm{d}s, \tag{7.30}$$

and $\underline{E_r} \in L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0};\mathbb{R}^3)$ depending only on $\mathcal{S}_0$ and Ω.

We now compute each term thanks to Lamb's lemma. More precisely we establish the following equalities:

$$\underline{\mathsf{E}}^{-1} = 0, \tag{7.31}$$

$$\mathcal{R}(\vartheta)\underline{\mathsf{E}}^{0}(q) = \mathsf{E}^{0}(q), \tag{7.32}$$

$$\mathcal{R}(\vartheta)\underline{\mathsf{E}}^{1}_a(q) = \mathsf{E}^{1}_a(q), \tag{7.33}$$

$$\mathcal{R}(\vartheta)\underline{\mathsf{E}}^{1}_b(q) = \mathsf{E}^{1}_b(q) + \mathsf{E}^{1}_c(q). \tag{7.34}$$

The proof is then concluded after observing that $\mathcal{R}(\vartheta)\underline{E_r}$ is also in $L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0};\mathbb{R}^3)$ and depends only on $\mathcal{S}_0$ and Ω.

In order to simplify the notations we omit to write the dependence on q except if this dependence reduces on ϑ or h. Similarly we omit to write that the functions $\boldsymbol{K}$, its coordinates K_j and the vector fields ξ_j, which appear thanks to Lamb's lemma, are evaluated at $q=0$.

Proof of (7.31)**. Computation of** $\underline{\mathsf{E}}^{-1}$. We use Lemma 22 with $u = v = \nabla^\perp\psi^{-1}_{\cancel{\partial\Omega}}$ and observe that $\nabla^\perp\psi^{-1}_{\cancel{\partial\Omega}}$ is tangent to $\mathcal{S}_0$ to obtain (7.31).

Proof of (7.32)**. Computation of** $\underline{\mathsf{E}}^{0}$. We observe that

$$\nabla^\perp\psi^{-1}_{\cancel{\partial\Omega}} = -\frac{\partial\psi^{-1}_{\cancel{\partial\Omega}}}{\partial n}\tau \text{ on } \partial\mathcal{S}_0, \tag{7.35}$$

that

$$\tau\cdot\nabla^\perp\psi^{0}_{\cancel{\partial\Omega}} = -\frac{\partial\psi^{0}_{\cancel{\partial\Omega}}}{\partial n} \text{ on } \partial\mathcal{S}_0, \tag{7.36}$$

so that, for $j=1,2,3$,

$$-\int_{\partial\mathcal{S}_0}\frac{\partial\psi^{-1}_{\cancel{\partial\Omega}}}{\partial n}\cdot\frac{\partial\psi^{0}_{\cancel{\partial\Omega}}}{\partial n}K_j\mathrm{d}s = -\int_{\partial\mathcal{S}_0}\nabla\psi^{-1}_{\cancel{\partial\Omega}}\cdot\nabla\psi^{0}_{\cancel{\partial\Omega}}K_j,\mathrm{d}s$$

and we use Lemma 22 with $(u,v) = (\nabla^\perp\psi^{-1}_{\cancel{\partial\Omega}}, \nabla^\perp\psi^{0}_{\cancel{\partial\Omega}})$ to obtain, still for $j=1,2,3$,

$$-\int_{\partial\mathcal{S}_0}\frac{\partial\psi^{-1}_{\cancel{\partial\Omega}}}{\partial n}\cdot\frac{\partial\psi^{0}_{\cancel{\partial\Omega}}}{\partial n}K_j\mathrm{d}s = -\int_{\partial\mathcal{S}_0}(\xi_j\cdot\nabla^\perp\psi^{-1}_{\cancel{\partial\Omega}})(n\cdot\nabla^\perp\psi^{0}_{\cancel{\partial\Omega}})\mathrm{d}s.$$

Then we use again (7.35) and observe that applying the tangential derivative to (5.17b), taking (5.15) into account, yields

$$n\cdot\nabla^\perp\psi^0_{\partial\Omega}=\frac{\partial\psi^0_{\partial\Omega}}{\partial\tau}=R(\vartheta)^t\,u_\Omega(h)^\perp\cdot\tau \text{ on } \partial\mathcal{S}_0. \tag{7.37}$$

Thus

$$-\int_{\partial\mathcal{S}_0}\frac{\partial\psi^{-1}_{\partial\Omega}}{\partial n}\cdot\frac{\partial\psi^0_{\partial\Omega}}{\partial n}K_j\mathrm{d}s=\int_{\partial\mathcal{S}_0}\frac{\partial\psi^{-1}_{\partial\Omega}}{\partial n}(\xi_j\cdot\tau)(R(\vartheta)^t\,u_\Omega(h)^\perp\cdot\tau)\mathrm{d}s. \tag{7.38}$$

On the other hand, we have:

$$\begin{aligned}\int_{\partial\mathcal{S}_0}\frac{\partial\psi^{-1}_{\partial\Omega}}{\partial n}\big(R(\vartheta)^t\,u_\Omega(h)\cdot\tau\big)\boldsymbol{K}\mathrm{d}s&=\Big(\int_{\partial\mathcal{S}_0}\frac{\partial\psi^{-1}_{\partial\Omega}}{\partial n}\big(R(\vartheta)^t\,u_\Omega(h)\cdot\tau\big)(\xi_j\cdot n)\mathrm{d}s\Big)_{j=1,2,3}\\&=\Big(\int_{\partial\mathcal{S}_0}\frac{\partial\psi^{-1}_{\partial\Omega}}{\partial n}(\xi_j\cdot n)\big(R(\vartheta)^t\,u_\Omega(h)^\perp\cdot n\big)\mathrm{d}s\Big)_{j=1,2,3}.\end{aligned} \tag{7.39}$$

Thus, combining (7.27), (7.38), (7.39), and then using (1.12d), (2.16e), (2.17) and (2.15), we get

$$\mathcal{R}(\vartheta)\underline{\mathsf{E}}^0=\mathcal{R}(\vartheta)\Big(\int_{\partial\mathcal{S}_0}\frac{\partial\psi^{-1}_{\partial\Omega}}{\partial n}\big(\xi_j\cdot R(\vartheta)^t\,u_\Omega(h)^\perp\big)\mathrm{d}s\Big)_j=-\begin{pmatrix}u_\Omega(h)\cdot\zeta_\vartheta\\ u_\Omega(h)^\perp\end{pmatrix}=\mathsf{E}^0. \tag{7.40}$$

This concludes the proof of (7.32).

Proof of (7.33)**. Computation of** $\underline{\mathsf{E}}^1_a$. We start with expanding the square in (7.29), to get

$$\underline{\mathsf{E}}^1_a=\underline{\mathsf{E}}^{1,1}_a+\int_{\partial\mathcal{S}_0}\frac{\partial\psi^0_{\partial\Omega}}{\partial n}\big(R(\vartheta)^t\,u_\Omega(h)\cdot\tau\big)\boldsymbol{K}\mathrm{d}s-\frac12\int_{\partial\mathcal{S}_0}\big|R(\vartheta)^t\,u_\Omega(h)\cdot\tau\big|^2\boldsymbol{K}\mathrm{d}s, \tag{7.41}$$

with

$$\underline{\mathsf{E}}^{1,1}_a=-\frac12\int_{\partial\mathcal{S}_0}\Big|\frac{\partial\psi^0_{\partial\Omega}}{\partial n}\Big|^2\boldsymbol{K}\mathrm{d}s=-\frac12\int_{\partial\mathcal{S}_0}\big|\nabla\psi^0_{\partial\Omega}\big|^2\boldsymbol{K}\mathrm{d}s+\frac12\int_{\partial\mathcal{S}_0}\Big|\frac{\partial\psi^0_{\partial\Omega}}{\partial\tau}\Big|^2\boldsymbol{K}\mathrm{d}s.$$

We apply Lemma 22 with $u=v=\nabla^\perp\psi^0_{\partial\Omega}$ to get

$$\frac12\int_{\partial\mathcal{S}_0}\big|\nabla\psi^0_{\partial\Omega}\big|^2\boldsymbol{K}\mathrm{d}s=\Big(\int_{\partial\mathcal{S}_0}\big(\nabla^\perp\psi^0_{\partial\Omega}\cdot n\big)\big(\nabla^\perp\psi^0_{\partial\Omega}\cdot\xi_j\big)\mathrm{d}s\Big)_{j=1,2,3}$$

Let us denote by $\underline{\mathsf{E}}^{1,1}_{a,j}$, $j=1,2,3$, the coordinates of the vector $\underline{\mathsf{E}}^{1,1}_a$. We use (7.37) to get

$$\underline{\mathsf{E}}^{1,1}_{a,j}=-\int_{\partial\mathcal{S}_0}\big(\xi_j\cdot\nabla^\perp\psi^0_{\partial\Omega}\big)\big(R(\vartheta)^t\,u_\Omega(h)^\perp\cdot\tau\big)\mathrm{d}s+\frac12\int_{\partial\mathcal{S}_0}\big(R(\vartheta)^t\,u_\Omega(h)^\perp\cdot\tau\big)^2K_j\,\mathrm{d}s.$$

Then we decompose $\xi_j\cdot\nabla^\perp\psi^0_{\partial\Omega}$ in normal and tangential parts and use again (7.37) to obtain:

$$\underline{\mathsf{E}}^{1,1}_{a,j}=\int_{\partial\mathcal{S}_0}\frac{\partial\psi^0_{\partial\Omega}}{\partial n}\big(R(\vartheta)^t\,u_\Omega(h)^\perp\cdot\tau\big)(\xi_j\cdot\tau)\,\mathrm{d}s-\frac12\int_{\partial\mathcal{S}_0}\big(R(\vartheta)^t\,u_\Omega(h)^\perp\cdot\tau\big)^2K_j\,\mathrm{d}s.$$

Now we plug this expression of $\underline{\mathsf{E}}^{1,1}_a$ into (7.41) to get

$$\begin{aligned}\underline{\mathsf{E}}^1_{a,j}=-\frac12\int_{\partial\mathcal{S}_0}\big(R(\vartheta)^t\,u_\Omega(h)^\perp\big)^2K_j\,\mathrm{d}s&+\int_{\partial\mathcal{S}_0}\frac{\partial\psi^0_{\partial\Omega}}{\partial n}\big(R(\vartheta)^t\,u_\Omega(h)\cdot\tau\big)K_j\mathrm{d}s\\&+\int_{\partial\mathcal{S}_0}\frac{\partial\psi^0_{\partial\Omega}}{\partial n}\big(R(\vartheta)^t\,u_\Omega(h)^\perp\cdot\tau\big)(\xi_j\cdot\tau)\mathrm{d}s.\end{aligned}$$

We observe that the first term in the right hand side vanishes and we combine the two other ones to get

$$\underline{\mathsf{E}}^1_{a,j} = \int_{\partial\mathcal{S}_0} \frac{\partial\psi^0_{\cancel{\partial\Omega}}}{\partial n}\big(R(\vartheta)^t\, u_\Omega(h)^\perp\big)\cdot\xi_j\,\mathrm{d}s.$$

Using (5.17c) we infer

$$\underline{\mathsf{E}}^1_{a,j} = 0 \text{ for } j = 2,3. \tag{7.42}$$

Now for $j=1$, we start with observing that

$$\underline{\mathsf{E}}^1_{a,1} = \big(R(\vartheta)^t\, u_\Omega(h)^\perp\big)\cdot\int_{\partial\mathcal{S}_0}\frac{\partial\psi^0_{\cancel{\partial\Omega}}}{\partial n}x^\perp\,\mathrm{d}s. \tag{7.43}$$

Then, combining (7.43) and (7.16), we obtain:

$$\underline{\mathsf{E}}^1_{a,1} = \big(R(\vartheta)^t\, u_\Omega(h)^\perp\big)\cdot\overline{M}R(\vartheta)^t\, u_\Omega(h)^\perp,$$

with $\overline{M}$ given by (7.15).

Using (7.22) and recalling the definition of $M^\dagger_\vartheta$ in (7.8), we get that

$$\underline{\mathsf{E}}^1_{a,1} = u_\Omega(h)^\perp M^\dagger_\vartheta\, u_\Omega(h)^\perp. \tag{7.44}$$

Gathering (7.42) and (7.44) we obtain (7.33).

Proof of (7.34)**. Computation of $\underline{\mathsf{E}}^1_b$.** We start with splitting $\underline{\mathsf{E}}^1_b$ into two parts as follows:

$$\underline{\mathsf{E}}^1_b = -\int_{\partial\mathcal{S}_0}\frac{\partial\psi^{-1}_{\cancel{\partial\Omega}}}{\partial n}\frac{\partial\psi^1_{\cancel{\partial\Omega}}}{\partial n}\boldsymbol{K}\mathrm{d}s + \int_{\partial\mathcal{S}_0}\frac{\partial\psi^{-1}_{\cancel{\partial\Omega}}}{\partial n}\frac{\partial P^1}{\partial n}\boldsymbol{K}\mathrm{d}s.$$

Using (7.35) and (7.36), the first term of the right hand side above is equal to

$$-\int_{\partial\mathcal{S}_0}\nabla^\perp\psi^{-1}_{\cancel{\partial\Omega}}\cdot\nabla^\perp\psi^1_{\cancel{\partial\Omega}}\boldsymbol{K}\,\mathrm{d}s.$$

We denote $\underline{\mathsf{E}}^1_{b,j}$, $j=1,2,3$, the coordinates of $\underline{\mathsf{E}}^1_b$. We apply Lemma 22 with $u=\nabla^\perp\psi^{-1}_{\cancel{\partial\Omega}}$ and $v=\nabla^\perp\psi^1_{\cancel{\partial\Omega}}$ for any $j=1,2,3$, to get

$$\underline{\mathsf{E}}^1_{b,j} = -\int_{\partial\mathcal{S}_0}\frac{\partial\psi^1_{\cancel{\partial\Omega}}}{\partial\tau}\xi_j\cdot\nabla^\perp\psi^{-1}_{\cancel{\partial\Omega}}\,\mathrm{d}s + \int_{\partial\mathcal{S}_0}\frac{\partial\psi^{-1}_{\cancel{\partial\Omega}}}{\partial n}\frac{\partial P^1}{\partial n}K_j\mathrm{d}s.$$

We now use that, on $\partial\mathcal{S}_0$,

$$\xi_j\cdot\nabla^\perp\psi^{-1}_{\cancel{\partial\Omega}} = -\frac{\partial\psi^{-1}_{\cancel{\partial\Omega}}}{\partial n}\xi_j\cdot\tau \quad\text{and}\quad \frac{\partial\psi^1_{\cancel{\partial\Omega}}}{\partial\tau} = \frac{\partial P^1}{\partial\tau},$$

the last identity being a consequence of (5.25b), to deduce

$$\underline{\mathsf{E}}^1_{b,j} = \int_{\partial\mathcal{S}_0}\frac{\partial\psi^{-1}_{\cancel{\partial\Omega}}}{\partial n}\xi_j\cdot\nabla P^1\mathrm{d}s. \tag{7.45}$$

Using the expression of P^1 in (5.22), we obtain

$$\underline{\mathsf{E}}^1_{b,j} = -\langle D^2_x\psi^0_{\cancel{\partial\mathcal{S}}}(h,h), R(\vartheta)A^1_jR(\vartheta)^t\rangle_{\mathbb{R}^{2\times2}} - D_x\psi^1_{\cancel{\partial\mathcal{S}}}(q,h)\cdot R(\vartheta)A^2_j, \tag{7.46}$$

where

$$A^1_j := \int_{\partial\mathcal{S}_0}\frac{\partial\psi^{-1}_{\cancel{\partial\Omega}}}{\partial n}x\otimes\xi_j\,\mathrm{d}s \quad\text{and}\quad A^2_j := \int_{\partial\mathcal{S}_0}\frac{\partial\psi^{-1}_{\cancel{\partial\Omega}}}{\partial n}\xi_j\,\mathrm{d}s.$$

• We start with the case $j=1$. Consider the first term in the right hand side of (7.46). We decompose A_1^1 into $A_1^1 = \sigma - \zeta \otimes \zeta^\perp$, where σ is given in (4.3) and we observe that, since $D_x^2 \psi^0_{\partial\mathcal{S}}(h,h)$ is symmetric,

$$\langle D_x^2 \psi^0_{\partial\mathcal{S}}(h,h), R(\vartheta)\sigma R(\vartheta)^t\rangle_{\mathbb{R}^{2\times 2}} = \langle D_x^2 \psi^0_{\partial\mathcal{S}}(h,h), R(\vartheta)\sigma^s R(\vartheta)^t\rangle$$

where σ^s is the symmetric part of σ defined in (4.4).

Then, using that σ^s is a traceless symmetric 2×2 matrix we get

$$\langle D_x^2 \psi^0_{\partial\mathcal{S}}(h,h), R(\vartheta)\sigma R(\vartheta)^t\rangle_{\mathbb{R}^{2\times 2}} = \langle D_x^2 \psi^0_{\partial\mathcal{S}}(h,h), R(-2\vartheta)\sigma^s\rangle = -\mathsf{E}^1_{b,1}(q),$$

where $\mathsf{E}^1_{b,1}(q)$ denotes the first coordinate of the vector field $\mathsf{E}^1_b(q)$ defined in (4.5). Therefore we obtain for $j=1$,

$$-\langle D_x^2 \psi^0_{\partial\mathcal{S}}(h,h), R(\vartheta)A^1_j R(\vartheta)^t\rangle_{\mathbb{R}^{2\times 2}} = \mathsf{E}^1_{b,1}(q) + \langle D_x^2 \psi^0_{\partial\mathcal{S}}(h,h), \zeta_\vartheta \otimes \zeta_\vartheta^\perp\rangle_{\mathbb{R}^{2\times 2}}.$$

Concerning the second term in the right hand side of (7.46), we use $A_1^2 = -\zeta^\perp$ (see (2.17)) to get that for $j=1$,

$$-D_x \psi^1_{\partial\mathcal{S}}(q,h)\cdot R(\vartheta)A_j^2 = D_x \psi^1_{\partial\mathcal{S}}(q,h)\cdot \zeta_\vartheta^\perp.$$

Thus

$$\underline{\mathsf{E}}^1_{b,1} = \mathsf{E}^1_{b,1}(q) + \langle D_x^2 \psi^0_{\partial\mathcal{S}}(h,h), \zeta_\vartheta \otimes \zeta_\vartheta^\perp\rangle_{\mathbb{R}^{2\times 2}} + D_x \psi^1_{\partial\mathcal{S}}(q,h)\cdot \zeta_\vartheta^\perp.$$

Let us now express the corrector velocity $u_c(q)$ defined in (3.18) thanks to the functions $\psi^0_{\partial\mathcal{S}}(h,h)$ and $\psi^1_{\partial\mathcal{S}}(q,h)$.

Lemma 29. *For any $q=(\vartheta,h)$ in $\Omega\times\mathbb{R}$,*

$$u_c(q) = \Big(D_x^2 \psi^0_{\partial\mathcal{S}}(h,h)\cdot\zeta_\vartheta + D_x \psi^1_{\partial\mathcal{S}}(q,h)\Big)^\perp. \tag{7.47}$$

Proof of Lemma 29. We first recall from (5.48) that for any $q=(\vartheta,h)$ in $\Omega\times\mathbb{R}$, $\psi_c(\vartheta,h) = D_x\psi^0_{\partial\mathcal{S}}(h,h)\cdot \zeta_\vartheta$. Hence from the definition of $u_c(q)$ in (3.18) we deduce that

$$u_c(q) = \Big(D_x^2 \psi^0_{\partial\mathcal{S}}(h,h)\cdot\zeta_\vartheta + D_{xh}^2 \psi^0_{\partial\mathcal{S}}(h,h)\cdot\zeta_\vartheta\Big)^\perp,$$

which yields (7.47) thanks to (5.19). □

Hence we obtain

$$\underline{\mathsf{E}}^1_{b,1} = \mathsf{E}^1_{b,1}(q) - \zeta_\vartheta\cdot u_c(q). \tag{7.48}$$

• On the other hand, for $j=2$ or 3, we have $A^1_j = -\zeta\otimes\xi_j$ and $A^2_j = -\xi_j$, and therefore

$$\begin{aligned}\underline{\mathsf{E}}^1_{b,j} &= \langle D_x^2 \psi^0_{\partial\mathcal{S}}(h,h), R(\vartheta)(\zeta\otimes\xi_j)R(\vartheta)^t\rangle_{\mathbb{R}^{2\times 2}} + D\psi^1_{\partial\mathcal{S}}(q,h)\cdot R(\vartheta)\xi_j \\ &= \Big(D_x^2 \psi^0_{\partial\mathcal{S}}(h,h)\cdot\zeta_\vartheta + D\psi^1_{\partial\mathcal{S}}(q,h)\Big)\cdot R(\vartheta)\xi_j \\ &= -R(\vartheta)^t\, u_c(q)^\perp\cdot\xi_j.\end{aligned}$$

Thus

$$R(\vartheta)(\underline{\mathsf{E}}^1_{b,1})_{j=2,3} = -u_c(q)^\perp. \tag{7.49}$$

Gathering (7.48) and (7.49) entails (7.34). This ends the proof of Proposition 12 □

7.4. **Expansion of B^ε.** We now tackle the expansion of B^ε which is given, for $(\varepsilon, q) \in \mathfrak{Q}$, by

$$B^\varepsilon(q) := \int_{\partial \mathcal{S}^\varepsilon(q)} \frac{\partial \psi^\varepsilon}{\partial n}(q,\cdot) \left(\boldsymbol{K}^\varepsilon(q,\cdot) \times \frac{\partial \boldsymbol{\varphi}^\varepsilon}{\partial \tau}(q,\cdot) \right) \mathrm{d}s.$$

This formula is the counterpart of (1.15a) for a body of size ε. Let us recall that the Kirchhoff's potentials $\boldsymbol{\varphi}^\varepsilon$ are defined in (5.28)- (5.29).

The expansion that we obtain for $B^\varepsilon(q)$ is given in the following statement.

Proposition 13. *Let $\delta > 0$. There exists $\varepsilon_0 \in (0,1)$ and a function $B_r \in L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0}; \mathbb{R}^3)$ depending only on $\mathcal{S}_0$ and Ω, such that for any (ε, q) in $\mathfrak{Q}_{\delta,\varepsilon_0}$,*

$$B^\varepsilon(q) = \varepsilon I_\varepsilon^{-1} \Big(B_{\cancel{\partial\Omega},\vartheta} + \varepsilon \mathsf{B}^1(q) + \varepsilon^2 B_r(\varepsilon, q) \Big), \tag{7.50}$$

with $B_{\cancel{\partial\Omega},\vartheta}$ as in (2.19) *and*

$$\mathsf{B}^1(q) := \begin{pmatrix} 0 \\ -2M_\vartheta^\dagger u_\Omega(h)^\perp \end{pmatrix}.$$

Proof of Proposition 13. We proceed as in the proof of Proposition 12. Let us state the following formula which is useful several times in the sequel:

$$\text{for any } p_a := (\omega_a, \ell_a),\ p_b := (\omega_b, \ell_b) \ \text{ in } \ \mathbb{R} \times \mathbb{R}^2, \quad \varepsilon\, p_a \times p_b = I_\varepsilon \Big((I_\varepsilon p_a) \times (I_\varepsilon p_b) \Big). \tag{7.51}$$

This formula easily follows from (2.20).

By a change of variable, using (7.51) and again (6.2), we get

$$B^\varepsilon(q) = \varepsilon I_\varepsilon^{-1} \mathcal{R}(\vartheta) \int_{\partial \mathcal{S}_0} \frac{\partial \psi^\varepsilon}{\partial n}(q, \varepsilon R(\vartheta)\cdot + h) \left(\boldsymbol{K}(0,\cdot) \times \mathcal{R}(\vartheta)^t \, \frac{\partial \boldsymbol{\varphi}^\varepsilon}{\partial \tau}(q, \varepsilon R(\vartheta)\cdot + h) \right) \mathrm{d}s.$$

Now let $\delta > 0$. We use (5.26) and (5.31) to obtain that there exists $\varepsilon_0 \in (0,1)$ such that for any (ε, q) in $\mathfrak{Q}_{\delta,\varepsilon_0}$,

$$B^\varepsilon(q) = \varepsilon I_\varepsilon^{-1} \mathcal{R}(\vartheta) \Big(\underline{\mathsf{B}}^0 + \varepsilon \underline{\mathsf{B}}^1(q) + \varepsilon^2 B_r(\varepsilon, q) \Big),$$

with

$$\underline{\mathsf{B}}^0 := \int_{\partial \mathcal{S}_0} \frac{\partial \psi^{-1}_{\cancel{\partial\Omega}}}{\partial n} \left(\boldsymbol{K}(0,\cdot) \times \frac{\partial \boldsymbol{\varphi}_{\cancel{\partial\Omega}}}{\partial \tau} \right) \mathrm{d}s,$$

$$\underline{\mathsf{B}}^1(q) := \int_{\partial \mathcal{S}_0} \left(\frac{\partial \psi^0_{\cancel{\partial\Omega}}}{\partial n} - R(\vartheta)^t \, u_\Omega(h) \cdot \tau \right) \left(\boldsymbol{K}(0,\cdot) \times \frac{\partial \boldsymbol{\varphi}_{\cancel{\partial\Omega}}}{\partial \tau} \right) \mathrm{d}s,$$

and $B_r \in L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0}; \mathbb{R}^3)$ depending only on $\mathcal{S}_0$ and Ω.

We now compute each term thanks to Lamb's lemma. More precisely we will prove the following equalities:

$$\mathcal{R}(\vartheta) \underline{\mathsf{B}}^0 = B_{\cancel{\partial\Omega},\vartheta}, \tag{7.52}$$

$$\mathcal{R}(\vartheta) \underline{\mathsf{B}}^1 = \mathsf{B}^1. \tag{7.53}$$

As in the proof of Proposition 13 we will omit to write the dependence on q, except if this dependence reduces to a dependence on ϑ or h, and it will be understood that the functions $\boldsymbol{K}$, its coordinates K_j and the vector fields ξ_j are evaluated at $q = 0$.

Proof of (7.52)**. Computation of** $\underline{\mathsf{B}}^0$. Let us denote by $\underline{\mathsf{B}}^0_j$, for $j = 1, 2, 3$, the coordinates of $\underline{\mathsf{B}}^0$. We have

$$\begin{aligned}\underline{\mathsf{B}}^0_1 &= \int_{\partial\mathcal{S}_0} \frac{\partial\psi^{-1}_{\partial\Omega}}{\partial n}\frac{\partial\varphi_{\partial\Omega,3}}{\partial\tau} K_2 \,\mathrm{d}s - \int_{\partial\mathcal{S}_0} \frac{\partial\psi^{-1}_{\partial\Omega}}{\partial n}\frac{\partial\varphi_{\partial\Omega,2}}{\partial\tau} K_3 \,\mathrm{d}s \\ &= -\int_{\partial\mathcal{S}_0} \nabla^\perp\psi^{-1}_{\partial\Omega}\cdot\nabla\varphi_{\partial\Omega,3}K_2\,\mathrm{d}s + \int_{\partial\mathcal{S}_0} \nabla^\perp\psi^{-1}_{\partial\Omega}\cdot\nabla\varphi_{\partial\Omega,2}K_3\,\mathrm{d}s,\end{aligned}$$

using (7.35). Then we use Lemma 22 with $(u, v) = (\nabla^\perp\psi^{-1}_{\partial\Omega}, \nabla\varphi_{\partial\Omega,2})$ and $(u, v) = (\nabla^\perp\psi^{-1}_{\partial\Omega}, \nabla\varphi_{\partial\Omega,3})$, and again (2.16e) and (7.35) to obtain

$$\underline{\mathsf{B}}^0_1 = \int_{\partial\mathcal{S}_0} \frac{\partial\psi^{-1}_{\partial\Omega}}{\partial n}\Big((\tau\cdot\xi_2)(n\cdot\xi_3) - (\tau\cdot\xi_3)(n\cdot\xi_2)\Big)\,\mathrm{d}s = -1.$$

Now, proceeding in the same way, we get

$$\begin{aligned}\underline{\mathsf{B}}^0_2 &= \int_{\partial\mathcal{S}_0} \frac{\partial\psi^{-1}_{\partial\Omega}}{\partial n}\frac{\partial\varphi_{\partial\Omega,1}}{\partial\tau} K_3 \,\mathrm{d}s - \int_{\partial\mathcal{S}_0} \frac{\partial\psi^{-1}_{\partial\Omega}}{\partial n}\frac{\partial\varphi_{\partial\Omega,3}}{\partial\tau} K_1 \,\mathrm{d}s \\ &= -\int_{\partial\mathcal{S}_0} \nabla^\perp\psi^{-1}_{\partial\Omega}\cdot\nabla\varphi_{\partial\Omega,1}K_3\,\mathrm{d}s + \int_{\partial\mathcal{S}_0} \nabla^\perp\psi^{-1}_{\partial\Omega}\cdot\nabla\varphi_{\partial\Omega,3}K_1\,\mathrm{d}s \\ &= \int_{\partial\mathcal{S}_0} \frac{\partial\psi^{-1}_{\partial\Omega}}{\partial n}\Big((\tau\cdot\xi_3)(n\cdot\xi_1) - (\tau\cdot\xi_1)(n\cdot\xi_3)\Big)\,\mathrm{d}s \\ &= -\int_{\partial\mathcal{S}_0} \frac{\partial\psi^{-1}_{\partial\Omega}}{\partial n}\xi_1\cdot\xi_2\,\mathrm{d}s \\ &= \zeta^\perp\cdot\xi_2,\end{aligned}$$

thanks to (2.17).

Proceeding in the same way we also have $\underline{\mathsf{B}}^0_3 = \zeta^\perp\cdot\xi_3$. This entails (7.52).

Proof of (7.53)**. Computation of** $\underline{\mathsf{B}}^1$. Let us start with the first coordinate $\underline{\mathsf{B}}^1_1$ of $\underline{\mathsf{B}}^1$, that is:

$$\begin{aligned}\underline{\mathsf{B}}^1_1 &= -\int_{\partial\mathcal{S}_0}\left(\frac{\partial\psi^0_{\partial\Omega}}{\partial n} - R(\vartheta)^t\,u_\Omega(h)\cdot\tau\right)\left(\frac{\partial\varphi_{\partial\Omega,2}}{\partial\tau}K_3 - \frac{\partial\varphi_{\partial\Omega,3}}{\partial\tau}K_2\right)\,\mathrm{d}s \\ &= \underline{\mathsf{B}}^1_{1,a} + \underline{\mathsf{B}}^1_{1,b} + \underline{\mathsf{B}}^1_{1,c},\end{aligned}$$

with

$$\begin{aligned}\underline{\mathsf{B}}^1_{1,a} &:= -\int_{\partial\mathcal{S}_0} \frac{\partial\psi^0_{\partial\Omega}}{\partial n}\frac{\partial\varphi_{\partial\Omega,2}}{\partial\tau}K_3\,\mathrm{d}s, \\ \underline{\mathsf{B}}^1_{1,b} &:= \int_{\partial\mathcal{S}_0} \frac{\partial\psi^0_{\partial\Omega}}{\partial n}\frac{\partial\varphi_{\partial\Omega,3}}{\partial\tau}K_2\,\mathrm{d}s, \\ \underline{\mathsf{B}}^1_{1,c} &:= \int_{\partial\mathcal{S}_0} R(\vartheta)^t\,u_\Omega(h)\cdot\left(\frac{\partial\varphi_{\partial\Omega,2}}{\partial\tau}K_3 - \frac{\partial\varphi_{\partial\Omega,3}}{\partial\tau}K_2\right)\tau\,\mathrm{d}s.\end{aligned}$$

We start with

$$\underline{\mathsf{B}}^1_{1,a} = \int_{\partial\mathcal{S}_0}\nabla^\perp\psi^0_{\partial\Omega}\cdot\nabla\varphi_{\partial\Omega,2}K_3\,\mathrm{d}s - \int_{\partial\mathcal{S}_0}\frac{\partial\psi^0_{\partial\Omega}}{\partial\tau}K_2K_3\,\mathrm{d}s.$$

We use Lemma 22 with $u = \nabla^\perp \psi^0_{\cancel{\partial\Omega}}$ and $v = \nabla \varphi_{\cancel{\partial\Omega},2}$ to obtain

$$\begin{aligned}\int_{\partial \mathcal{S}_0} \nabla^\perp \psi^0_{\cancel{\partial\Omega}} \cdot \nabla \varphi_{\cancel{\partial\Omega},2} K_3 \,\mathrm{d}s &= \int_{\partial \mathcal{S}_0} \big(\nabla^\perp \psi^0_{\cancel{\partial\Omega}} \cdot \xi_3\big)\big(\nabla \varphi_{\cancel{\partial\Omega},2} \cdot n\big) \,\mathrm{d}s \\ &\quad + \int_{\partial \mathcal{S}_0} \big(\nabla^\perp \psi^0_{\cancel{\partial\Omega}} \cdot n\big)\big(\nabla \varphi_{\cancel{\partial\Omega},2} \cdot \xi_3\big) \,\mathrm{d}s \\ &= \int_{\partial \mathcal{S}_0} \big(\frac{\partial \psi^0_{\cancel{\partial\Omega}}}{\partial \tau} K_3 - \frac{\partial \psi^0_{\cancel{\partial\Omega}}}{\partial n} \xi_3 \cdot \tau\big) K_2 \,\mathrm{d}s + \int_{\partial \mathcal{S}_0} \frac{\partial \psi^0_{\cancel{\partial\Omega}}}{\partial \tau} K_2 K_3 \,\mathrm{d}s \\ &\quad + \int_{\partial \mathcal{S}_0} \frac{\partial \psi^0_{\cancel{\partial\Omega}}}{\partial \tau} \frac{\partial \varphi_{\cancel{\partial\Omega},2}}{\partial \tau} (\xi_3 \cdot \tau) \,\mathrm{d}s.\end{aligned}$$

Therefore

$$\underline{\mathsf{B}}^1_{1,a} = \int_{\partial \mathcal{S}_0} \big(\frac{\partial \psi^0_{\cancel{\partial\Omega}}}{\partial \tau} K_3 - \frac{\partial \psi^0_{\cancel{\partial\Omega}}}{\partial n} \xi_3 \cdot \tau\big) K_2 \,\mathrm{d}s + \int_{\partial \mathcal{S}_0} \frac{\partial \psi^0_{\cancel{\partial\Omega}}}{\partial \tau} \frac{\partial \varphi_{\cancel{\partial\Omega},2}}{\partial \tau} (\xi_3 \cdot \tau) \,\mathrm{d}s.$$

By switching the indexes 2 and 3 we obtain

$$\underline{\mathsf{B}}^1_{1,b} = -\int_{\partial \mathcal{S}_0} \big(\frac{\partial \psi^0_{\cancel{\partial\Omega}}}{\partial \tau} K_2 - \frac{\partial \psi^0_{\cancel{\partial\Omega}}}{\partial n} \xi_2 \cdot \tau\big) K_3 \,\mathrm{d}s - \int_{\partial \mathcal{S}_0} \frac{\partial \psi^0_{\cancel{\partial\Omega}}}{\partial \tau} \frac{\partial \varphi_{\cancel{\partial\Omega},3}}{\partial \tau} (\xi_2 \cdot \tau) \,\mathrm{d}s.$$

We sum these two terms, observing that $-(\xi_3 \cdot \tau) K_2 + (\xi_2 \cdot \tau) K_3 = K_2^2 + K_3^2$ and

$$\int_{\partial \mathcal{S}_0} (K_2^2 + K_3^2) \frac{\partial \psi^0_{\cancel{\partial\Omega}}}{\partial n} \,\mathrm{d}s = \int_{\partial \mathcal{S}_0} \frac{\partial \psi^0_{\cancel{\partial\Omega}}}{\partial n} \,\mathrm{d}s = 0,$$

thanks to (5.17c), to get

$$\underline{\mathsf{B}}^1_{1,a} + \underline{\mathsf{B}}^1_{1,b} = \int_{\partial \mathcal{S}_0} \frac{\partial \psi^0_{\cancel{\partial\Omega}}}{\partial \tau} \Big(\frac{\partial \varphi_{\cancel{\partial\Omega},2}}{\partial \tau} (\xi_3 \cdot \tau) - \frac{\partial \varphi_{\cancel{\partial\Omega},3}}{\partial \tau} (\xi_2 \cdot \tau)\Big) \,\mathrm{d}s.$$

Now, using (7.37), we obtain

$$\begin{aligned}\underline{\mathsf{B}}^1_{1,a} + \underline{\mathsf{B}}^1_{1,b} &= -\int_{\partial \mathcal{S}_0} R(\vartheta)^t \, u_\Omega(h) \cdot \Big(\frac{\partial \varphi_{\cancel{\partial\Omega},2}}{\partial \tau} (\xi_3 \cdot \tau) - \frac{\partial \varphi_{\cancel{\partial\Omega},3}}{\partial \tau} (\xi_2 \cdot \tau)\Big) n \,\mathrm{d}s \\ &= R(\vartheta)^t \, u_\Omega(h) \cdot \int_{\partial \mathcal{S}_0} \Big(\frac{\partial \varphi_{\cancel{\partial\Omega},2}}{\partial \tau} (\xi_2 \cdot n) + \frac{\partial \varphi_{\cancel{\partial\Omega},3}}{\partial \tau} (\xi_3 \cdot n)\Big) n \,\mathrm{d}s. \qquad (7.54)\end{aligned}$$

On the other hand we observe that

$$\underline{\mathsf{B}}^1_{1,c} = R(\vartheta)^t \, u_\Omega(h) \cdot \int_{\partial \mathcal{S}_0} \Big(\frac{\partial \varphi_{\cancel{\partial\Omega},2}}{\partial \tau} (\xi_2 \cdot \tau) + \frac{\partial \varphi_{\cancel{\partial\Omega},3}}{\partial \tau} (\xi_3 \cdot \tau)\Big) \tau \,\mathrm{d}s. \qquad (7.55)$$

As a consequence, gathering (7.54) and (7.55) we get

$$\underline{\mathsf{B}}^1_{1} = R(\vartheta)^t \, u_\Omega(h) \cdot \int_{\partial \mathcal{S}_0} \Big[\frac{\partial \varphi_{\cancel{\partial\Omega},2}}{\partial \tau} \xi_2 + \frac{\partial \varphi_{\cancel{\partial\Omega},3}}{\partial \tau} \xi_3\Big] \,\mathrm{d}s = 0,$$

by integration by parts.

Let us now consider the second coordinate $\underline{\mathsf{B}}^1_2$ of $\underline{\mathsf{B}}^1$, that is:

$$\begin{aligned}\underline{\mathsf{B}}^1_2 &= \int_{\partial \mathcal{S}_0} \Big(\frac{\partial \psi^0_{\cancel{\partial\Omega}}}{\partial n} - R(\vartheta)^t \, u_\Omega(h) \cdot \tau\Big) \Big(\frac{\partial \varphi_{\cancel{\partial\Omega},1}}{\partial \tau} K_3 - \frac{\partial \varphi_{\cancel{\partial\Omega},3}}{\partial \tau} K_1\Big) \,\mathrm{d}s \\ &= \underline{\mathsf{B}}^1_{2,a} + \underline{\mathsf{B}}^1_{2,b} + \underline{\mathsf{B}}^1_{2,c},\end{aligned}$$

with

$$\underline{\mathsf{B}}^1_{2,a} := -\int_{\partial\mathcal{S}_0} \frac{\partial\psi^0_{\partial\Omega}}{\partial n}\frac{\partial\varphi_{\partial\Omega,3}}{\partial\tau} K_1 \,\mathrm{d}s,$$

$$\underline{\mathsf{B}}^1_{2,b} := \int_{\partial\mathcal{S}_0} \frac{\partial\psi^0_{\partial\Omega}}{\partial n}\frac{\partial\varphi_{\partial\Omega,1}}{\partial\tau} K_3 \,\mathrm{d}s,$$

$$\underline{\mathsf{B}}^1_{2,c} := \int_{\partial\mathcal{S}_0} R(\vartheta)^t\, u_\Omega(h)\cdot\Big(\frac{\partial\varphi_{\partial\Omega,3}}{\partial\tau}K_1 - \frac{\partial\varphi_{\partial\Omega,1}}{\partial\tau}K_3\Big)\tau \,\mathrm{d}s.$$

Proceeding as above with $\underline{\mathsf{B}}^1_{1,a}$ and $\underline{\mathsf{B}}^1_{1,b}$ we get

$$\underline{\mathsf{B}}^1_{2,a} = \int_{\partial\mathcal{S}_0} \big(\frac{\partial\psi^0_{\partial\Omega}}{\partial\tau}K_1 - \frac{\partial\psi^0_{\partial\Omega}}{\partial n}\xi_1\cdot\tau\big)K_3 \,\mathrm{d}s + \int_{\partial\mathcal{S}_0} \frac{\partial\psi^0_{\partial\Omega}}{\partial\tau}\frac{\partial\varphi_{\partial\Omega,3}}{\partial\tau}(\xi_1\cdot\tau)\,\mathrm{d}s,$$

$$\underline{\mathsf{B}}^1_{2,b} = -\int_{\partial\mathcal{S}_0} \big(\frac{\partial\psi^0_{\partial\Omega}}{\partial\tau}K_3 - \frac{\partial\psi^0_{\partial\Omega}}{\partial n}\xi_3\cdot\tau\big)K_1 \,\mathrm{d}s - \int_{\partial\mathcal{S}_0} \frac{\partial\psi^0_{\partial\Omega}}{\partial\tau}\frac{\partial\varphi_{\partial\Omega,1}}{\partial\tau}(\xi_3\cdot\tau)\,\mathrm{d}s.$$

We sum these two terms, observing that

$$(\xi_1\cdot\tau)K_3 - (\xi_3\cdot\tau)K_1 = x^\perp\cdot\xi_2, \tag{7.56}$$

to get

$$\underline{\mathsf{B}}^1_{2,a} + \underline{\mathsf{B}}^1_{2,b} = -\int_{\partial\mathcal{S}_0}(x^\perp\cdot\xi_2)\frac{\partial\psi^0_{\partial\Omega}}{\partial n}\,\mathrm{d}s + \int_{\partial\mathcal{S}_0}\frac{\partial\psi^0_{\partial\Omega}}{\partial\tau}\Big(\frac{\partial\varphi_{\partial\Omega,3}}{\partial\tau}(\xi_1\cdot\tau) - \frac{\partial\varphi_{\partial\Omega,1}}{\partial\tau}(\xi_3\cdot\tau)\Big)\,\mathrm{d}s. \tag{7.57}$$

Using (7.16) we obtain

$$\int_{\partial\mathcal{S}_0}(x^\perp\cdot\xi_2)\frac{\partial\psi^0_{\partial\Omega}}{\partial n}\,\mathrm{d}s = \overline{M}R(\vartheta)^t\, u_\Omega(h)^\perp\cdot\xi_2, \tag{7.58}$$

with $\overline{M}$ given by (7.15). We also use (7.37) to modify the second term in the right hand side of (7.57) and then get

$$\underline{\mathsf{B}}^1_{2,a} + \underline{\mathsf{B}}^1_{2,b} = -\overline{M}R(\vartheta)^t\, u_\Omega(h)^\perp\cdot\xi_2 - \int_{\partial\mathcal{S}_0} R(\vartheta)^t\, u_\Omega(h)\cdot\Big(\frac{\partial\varphi_{\partial\Omega,3}}{\partial\tau}(\xi_1\cdot\tau) - \frac{\partial\varphi_{\partial\Omega,1}}{\partial\tau}(\xi_3\cdot\tau)\Big)n\,\mathrm{d}s.$$

Adding $\underline{\mathsf{B}}^1_{2,c}$ to this we get

$$\underline{\mathsf{B}}^1_{2} = -\overline{M}R(\vartheta)^t\, u_\Omega(h)^\perp\cdot\xi_2 - R(\vartheta)^t\, u_\Omega(h)\cdot\underline{\mathsf{B}}^1_{2,d}, \tag{7.59}$$

with

$$\begin{aligned}\underline{\mathsf{B}}^1_{2,d} &:= \int_{\partial\mathcal{S}_0}\frac{\partial\varphi_{\partial\Omega,3}}{\partial\tau}\Big((\xi_1^\perp\cdot n)n + (\xi_1^\perp\cdot\tau)\tau\Big)\,\mathrm{d}s + \int_{\partial\mathcal{S}_0}\frac{\partial\varphi_{\partial\Omega,1}}{\partial\tau}\Big(-(\xi_3^\perp\cdot n)n - (\xi_3^\perp\cdot\tau)\tau\Big)\,\mathrm{d}s\\ &= \int_{\partial\mathcal{S}_0}\frac{\partial\varphi_{\partial\Omega,3}}{\partial\tau}\xi_1^\perp\,\mathrm{d}s - \int_{\partial\mathcal{S}_0}\frac{\partial\varphi_{\partial\Omega,1}}{\partial\tau}\xi_3^\perp\,\mathrm{d}s.\end{aligned}$$

Using an integration by parts we see that the second term of the right hand side above vanishes. Thus

$$\begin{aligned}R(\vartheta)^t\, u_\Omega(h)\cdot\underline{\mathsf{B}}^1_{2,d} &= -R(\vartheta)^t\, u_\Omega(h)^\perp\cdot\int_{\partial\mathcal{S}_0}\frac{\partial\varphi_{\partial\Omega,3}}{\partial\tau}x^\perp\,\mathrm{d}s\\ &= \overline{M}^t\, R(\vartheta)^t\, u_\Omega(h)^\perp\cdot\xi_2,\end{aligned} \tag{7.60}$$

thanks to (7.20).

Gathering (7.59) and (7.60) and using (7.22) we get

$$\underline{\mathsf{B}}^1_{2} = -2M^\dagger R(\vartheta)^t\, u_\Omega(h)^\perp\cdot\xi_2. \tag{7.61}$$

Proceeding in the same way for the third coordinate, using

$$(\xi_1 \cdot \tau)K_2 - (\xi_2 \cdot \tau)K_1 = -x^\perp \cdot \xi_3,$$

instead of (7.56) and (7.21) instead of (7.20) we get

$$\underline{\mathsf{B}}_3^1 = -2M^\dagger R(\vartheta)^t\, u_\Omega(h)^\perp \cdot \xi_3. \tag{7.62}$$

Combining (7.61) and (7.62) and recalling the definition of $M_\vartheta^\dagger$ in (7.8), we get

$$\mathcal{R}(\vartheta)\underline{\mathsf{B}}^1 = \begin{pmatrix} 0 \\ -2M_\vartheta^\dagger u_\Omega(h)^\perp \end{pmatrix} = \mathsf{B}^1,$$

which concludes the proof of (7.53) and hence of Proposition 13. □

7.5. **Proof of Proposition 5.** Let us now proceed to the proof of Proposition 5. We recall, see (4.2), that $H^\varepsilon(q,p)$ is defined, for $(\varepsilon,q,p) \in \mathfrak{Q} \times \mathbb{R}^3$, by

$$H^\varepsilon(q,p) := F^\varepsilon(q,p) - \langle \Gamma^\varepsilon(q), p, p\rangle.$$

Using the decomposition of the Christoffel symbols provided by Proposition 2 we infer that for $(\varepsilon,q,p) \in \mathfrak{Q} \times \mathbb{R}^3$,

$$H^\varepsilon(q,p) = \gamma^2 E^\varepsilon(q) + \gamma p \times B^\varepsilon(q) - \langle \Gamma_{\mathcal{S}}^\varepsilon(q), p, p\rangle - \langle \Gamma_{\partial\Omega}^\varepsilon(q), p, p\rangle.$$

We now plug the expansion of each term of the right hand side using the results above. Let us start with the term involving $B^\varepsilon(q)$. Let $\delta > 0$. Using (7.50) and (7.51) we obtain that there exists $\varepsilon_0 \in (0,1)$ such that for any (ε,q,p) in $\mathfrak{Q}_{\delta,\varepsilon_0} \times \mathbb{R}^3$,

$$p \times B^\varepsilon(q) = I_\varepsilon\Big(\hat{p} \times \big(B_{\cancel{\partial\Omega},\vartheta} + \varepsilon \mathsf{B}^1(q) + \varepsilon^2 B_r(\varepsilon,q)\big)\Big),$$

where $\hat{p} = I_\varepsilon p$.

Therefore, using now the expansions of the other terms, that is (6.7), (6.9) and (7.24), we get, reducing $\varepsilon_0 \in (0,1)$ if necessary, that for any (ε,q,p) in $\mathfrak{Q}_{\delta,\varepsilon_0} \times \mathbb{R}^3$,

$$I_\varepsilon^{-1} H^\varepsilon(q,p) = \Big(\gamma^2 \mathsf{E}^0(q) + \gamma\hat{p} \times B_{\cancel{\partial\Omega},\vartheta}\Big) + \varepsilon\Big(\gamma^2\mathsf{E}^1(q) + \gamma\hat{p}\times \mathsf{B}^1(q) - \langle \Gamma_{\cancel{\partial\Omega},\vartheta}, \hat{p}, \hat{p}\rangle\Big) + \varepsilon^2 H_r(\varepsilon,q,\tilde{p}), \tag{7.63}$$

where

$$H_r(\varepsilon,q,\hat{p}) = \gamma^2 E_r(\varepsilon,q) + \gamma\hat{p} \times B_r(\varepsilon,q) - \varepsilon\langle \Gamma_{\mathcal{S},r}(\varepsilon,q), \hat{p}, \hat{p}\rangle - \varepsilon\langle \Gamma_{\partial\Omega,r}(\varepsilon,q), \hat{p}, \hat{p}\rangle.$$

Then we observe that the zero order term in the right hand side of (7.63) can be recast as follows:

$$\gamma^2 \mathsf{E}^0(q) + \gamma\hat{p} \times B_{\cancel{\partial\Omega},\vartheta}(q) = F_{\cancel{\partial\Omega},\vartheta}(\tilde{p}). \tag{7.64}$$

Now, in order to deal with the subprincipal term of the right hand side of (7.63), let us state the following crucial lemma, where we consider the part $\mathsf{E}_a^1(q)$ defined in (7.23) of the decomposition (7.25) of the term $\mathsf{E}^1(q)$.

Lemma 30. *For any $q = (\vartheta, h) \in \mathbb{R} \times \Omega$, for any $p := (\omega, \ell) \in \mathbb{R}^3$,*

$$\gamma^2 \mathsf{E}_a^1(q) + \gamma\hat{p} \times \mathsf{B}^1(q) - \langle \Gamma_{\cancel{\partial\Omega},\vartheta}, \hat{p}, \hat{p}\rangle = -\langle \Gamma_{\cancel{\partial\Omega},\vartheta}, \tilde{p}, \tilde{p}\rangle, \tag{7.65}$$

where $\hat{p} = I_\varepsilon p$ and $\tilde{p} := (\hat{\omega}, \tilde{\ell})$, with $\hat{\omega} := \varepsilon\omega$ and $\tilde{\ell} := \ell - \gamma u_\Omega(h)$.

Proof of Lemma 30. Let $q = (\vartheta, h) \in \mathbb{R} \times \Omega$ and $p := (\omega, \ell) \in \mathbb{R}^3$ and $\hat{p} = I_\varepsilon p$. Using (7.65) and that for any $\vartheta \in \mathbb{R}$, $M_\vartheta^\dagger$ is symmetric, we obtain

$$\hat{p} \times \mathsf{B}^1(q) = \begin{pmatrix} -2\ell^\perp \cdot M_\vartheta^\dagger \, u_\Omega(h)^\perp \\ -2\tilde{\omega}^2 \big(M_\vartheta^\dagger \, u_\Omega(h)^\perp\big)^\perp \end{pmatrix} = \begin{pmatrix} -2\ell^\perp \cdot M_\vartheta^\dagger \, u_\Omega(h)^\perp \\ -2\tilde{\omega}^2 M_\vartheta^\dagger \, u_\Omega(h) \end{pmatrix}, \tag{7.66}$$

thanks to (7.11). We recall that, according to (7.12) and (7.23) we have:

$$\langle \Gamma_{\cancel{\partial\Omega},\vartheta}, \hat{p}, \hat{p} \rangle = \begin{pmatrix} -\ell^\perp \cdot M_\vartheta^\dagger \, \ell^\perp \\ \tilde{\omega}^2 \mu_\vartheta^\perp - 2\tilde{\omega} M_\vartheta^\dagger \, \ell \end{pmatrix} \quad \text{and} \quad \mathsf{E}_a^1(q) := \begin{pmatrix} u_\Omega(h)^\perp M_\vartheta^\dagger \, u_\Omega(h)^\perp \\ 0 \\ 0 \end{pmatrix}. \tag{7.67}$$

Now it suffices to combine (7.66) and (7.67) to deduce (7.65). □

As a consequence, we obtain:

$$\gamma^2 \mathsf{E}^1(q) + \gamma \hat{p} \times \mathsf{B}^1(q) - \langle \Gamma_{\cancel{\partial\Omega},\vartheta}, \hat{p}, \hat{p} \rangle = \mathsf{H}^1(q, \tilde{p}), \tag{7.68}$$

where $\tilde{p}$ is defined in (4.8) and $\mathsf{H}^1(q, \tilde{p})$ is given by (4.10).

Combining (7.63), (7.64) and (7.68) we get that Equation (4.9) holds true for all (ε, q, p) in $\mathfrak{Q}_{\delta,\varepsilon_0} \times \mathbb{R}^3$. Moreover $H_r \in L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0} \times \mathbb{R}^3; \mathbb{R}^3)$, depends on $\mathcal{S}_0$, γ and Ω and is weakly nonlinear in the sense of Definition 2. This concludes the proof of Proposition 5. □

8. Asymptotic energy estimates and passage to the limit

In this section, we prove Corollary 4 and Lemma 8 and establish the main results by passing to the limit as $\varepsilon \to 0^+$.

8.1. **Proof of Corollary 4.** First, according to Lemma 1 and reducing ε_0 and δ if necessary, $(\varepsilon, q^\varepsilon)$ belonging to $\mathfrak{Q}_{\delta,\varepsilon_0}$ implies that h^ε is in Ω_δ. We recall that the sets Ω_δ, for $\delta > 0$ are defined in (2.5).

Thanks to Corollary 3 we have, on $[0, T]$, for $\varepsilon \in (0, \varepsilon_0)$,

$$\breve{\mathcal{E}}^\varepsilon(q^\varepsilon, \hat{p}^\varepsilon) = \varepsilon^{\min(2,\alpha)} \, \tilde{\mathcal{E}}_\vartheta(\varepsilon, \hat{p}^\varepsilon) + \frac{1}{2}\varepsilon^4 M_r(\varepsilon, q^\varepsilon)\hat{p}^\varepsilon \cdot \hat{p}^\varepsilon - \gamma^2 \psi_\Omega(h^\varepsilon) + \varepsilon U_r(\varepsilon, q^\varepsilon) = \breve{\mathcal{E}}^\varepsilon(0, p_0).$$

Now, since $h_0 = 0 \in \Omega_\delta$ and $q(0) = 0 \in \mathcal{Q}_\delta$, we infer from Lemma 3 and Lemma 4 that the initial renormalized energy $\breve{\mathcal{E}}^\varepsilon(0, p_0)$, when ε runs into $(0, \varepsilon_0)$, is bounded by a constant $K_1 > 0$ depending only on

$$\mathcal{S}_0, \ \Omega, \ p_0, \ \gamma, \ m^1, \ \mathcal{J}^1, \ \delta. \tag{8.1}$$

Moreover, on $[0, T]$, for $\varepsilon \in (0, \varepsilon_0)$, $\gamma^2 \psi_\Omega(h^\varepsilon)$ is bounded by a constant $K_2 > 0$ depending only on (8.1).

Using furthermore that, according to Lemma 3, U_r is in $L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0}; \mathbb{R})$ we get that on $[0, T]$, for $\varepsilon \in (0, \varepsilon_0)$, $-\varepsilon U_r(\varepsilon, q^\varepsilon)$ is bounded by a constant $K_3 > 0$ depending only on (8.1).

On the other hand, thanks to Lemma 4 and Lemma 5, reducing again ε_0 if necessary, we have that on $[0, T]$, for $\varepsilon \in (0, \varepsilon_0)$,

$$\varepsilon^{\min(2,\alpha)} \, \tilde{\mathcal{E}}_\vartheta(\varepsilon, \hat{p}^\varepsilon) + \frac{1}{2}\varepsilon^4 M_r(\varepsilon, q^\varepsilon)\hat{p}^\varepsilon \cdot \hat{p}^\varepsilon \geqslant K_4 \Big(\varepsilon^{\min(1,\frac{\alpha}{2})} \, |\hat{p}^\varepsilon|_{\mathbb{R}^3}\Big)^2.$$

Corollary 4 then follows by choosing for instance $K = (K_4^{-1}(K_1 + K_2 + K_3))^{\frac{1}{2}}$. □

8.2. **Proof of Lemma 8.** Using the symmetry of the matrix $\tilde{M}_{\vartheta^\varepsilon}$, we get that the time derivative $(\tilde{\mathcal{E}}_{\vartheta^\varepsilon}(\varepsilon,\tilde{\tilde{p}}^\varepsilon))'$ of the modulated energy is

$$\frac{d}{dt}\tilde{\mathcal{E}}_{\vartheta^\varepsilon}(\varepsilon,\tilde{\tilde{p}}^\varepsilon) = \tilde{\tilde{p}}^\varepsilon \cdot \tilde{M}_{\vartheta^\varepsilon}(\varepsilon)\frac{d}{dt}\tilde{\tilde{p}}^\varepsilon + \frac{1}{2}\tilde{\tilde{p}}^\varepsilon \cdot \left(\frac{d}{dt}\tilde{M}_{\vartheta^\varepsilon}(\varepsilon)\right)\tilde{\tilde{p}}^\varepsilon. \tag{8.2}$$

Combining (8.2) and (3.23) we get

$$\varepsilon^{\min(2,\alpha)}\frac{d}{dt}\tilde{\mathcal{E}}_{\vartheta^\varepsilon}(\varepsilon,\tilde{\tilde{p}}^\varepsilon) = \tilde{\tilde{p}}^\varepsilon \cdot F_{\cancel{\partial\Omega},\vartheta^\varepsilon}(\tilde{\tilde{p}}^\varepsilon) + \tilde{\tilde{p}}^\varepsilon \cdot \left(\frac{1}{2}\varepsilon^{\min(2,\alpha)}\left(\frac{d}{dt}\tilde{M}_{\vartheta^\varepsilon}(\varepsilon)\right)\tilde{\tilde{p}}^\varepsilon - \varepsilon\langle\Gamma_{\cancel{\partial\Omega},\vartheta^\varepsilon},\tilde{\tilde{p}}^\varepsilon,\tilde{\tilde{p}}^\varepsilon\rangle\right) + \varepsilon\gamma^2\tilde{\tilde{p}}^\varepsilon \cdot \mathsf{E}_b^1(q^\varepsilon) + \varepsilon^{\min(2,\alpha)}\,\tilde{\tilde{p}}^\varepsilon \cdot \tilde{H}_r(\varepsilon,q^\varepsilon,\tilde{\tilde{p}}^\varepsilon). \tag{8.3}$$

Using that the force term is $F_{\cancel{\partial\Omega},\vartheta}(p)$ is gyroscopic in the sense of Definition 1 we get that the first term of the right hand side of the equation (8.3) vanishes.

Let us now show that the second term in the right hand side of the equation (8.3) vanishes as well. Going back to the definition of $\tilde{M}_\vartheta(\varepsilon)$ in Section 3.2 we obtain that

$$\varepsilon^2\frac{d}{dt}M_{\cancel{\partial\Omega},\vartheta^\varepsilon} = \varepsilon^{\min(2,\alpha)}\frac{d}{dt}\tilde{M}_{\vartheta^\varepsilon}(\varepsilon). \tag{8.4}$$

On the one hand we observe that

$$\frac{d}{dt}M_{\cancel{\partial\Omega},\vartheta^\varepsilon} = \frac{1}{\varepsilon}\frac{\partial M_{\cancel{\partial\Omega},\vartheta}}{\partial q}(\vartheta^\varepsilon)\cdot\tilde{\tilde{p}}^\varepsilon. \tag{8.5}$$

On the other hand we introduce, for any $\vartheta \in \mathbb{R}$, for any $p \in \mathbb{R}^3$, the matrix

$$S_{\cancel{\partial\Omega},\vartheta}(p) := \left(\sum_{1\leqslant i\leqslant 3}(\Gamma_{\cancel{\partial\Omega},\vartheta})^k_{i,j}p_i\right)_{1\leqslant k,j\leqslant 3},$$

so that

$$\langle\Gamma_{\cancel{\partial\Omega},\vartheta},p,p\rangle = S_{\cancel{\partial\Omega},\vartheta}(p)p.$$

Then, we observe that for any $\vartheta \in \mathbb{R}$, for any $p \in \mathbb{R}^3$,

$$\frac{1}{2}\frac{\partial M_{\cancel{\partial\Omega},\vartheta}}{\partial q}(\vartheta)\cdot p - S_{\cancel{\partial\Omega},\vartheta}(p) \text{ is skew-symmetric.} \tag{8.6}$$

This can be checked by using the explicit expression of $M_{\cancel{\partial\Omega},\cdot}(\vartheta)$ and $S_{\cancel{\partial\Omega},\vartheta}$. Another method, more theoretical, is given in the second proof of Proposition 1 in Subsection 9.6.2, see Lemma 43. Combining (8.5), (8.6) and (8.4) entails that the second term of the right hand side of the equation (8.3) vanishes. Therefore the equation (8.3) reduces to (3.25). □

8.3. **Proof of Lemma 9.** Let us recall that

$$R_0 := \max\{|x|,\ x \in \partial\mathcal{S}_0\}, \tag{8.7}$$

so that, whatever $t \geqslant 0$, $\vartheta(t) \in \mathbb{R}$ and $\varepsilon \in (0,1)$,

$$\mathcal{S}^\varepsilon(q^\varepsilon(t)) \subset \overline{B}(h^\varepsilon(t),\varepsilon R_0). \tag{8.8}$$

We introduce

$$\overline{\delta} := \frac{1}{4}d(0,\partial\Omega). \tag{8.9}$$

and introduce $\varepsilon_0 \in (0,1)$ (which may be reduced later) such that

$$\varepsilon_0 R_0 \leqslant \overline{\delta}, \tag{8.10}$$

$$\forall\varepsilon \in (0,\varepsilon_0],\quad d\big(\overline{B}(0,\varepsilon R_0),\partial\Omega\big) \geqslant \frac{3}{4}d(0,\partial\Omega). \tag{8.11}$$

We apply Corollary 5 with $\overline{\delta}$ defined in (8.9) and $T = 1$ to deduce that there exists $K > 0$ such that, reducing ε_0 if necessary, we have

$$|\ell^\varepsilon| \leqslant K, \text{ for all } t \in [0,1] \text{ for which } d\big(\mathcal{S}^\varepsilon(q^\varepsilon(t)), \partial\Omega\big) \geqslant \overline{\delta}. \tag{8.12}$$

We introduce

$$\overline{T} := \min\left(1, \frac{d(0,\partial\Omega)}{2K}\right),$$

and, for $\varepsilon \in (0, \varepsilon_0]$,

$$\mathcal{I}^\varepsilon = \left\{t \in [0,1] \ / \ \forall s \in [0,t], \ d\big(\overline{B}(h^\varepsilon(s), \varepsilon R_0), \partial\Omega\big) \geqslant \overline{\delta}\right\}.$$

The set $\mathcal{I}^\varepsilon$ is a closed interval containing 0, according to (8.11). Consider $\tilde{T}^\varepsilon := \max \mathcal{I}^\varepsilon$, and let us show that $\tilde{T}^\varepsilon \geqslant \overline{T}$. Of course, if $\tilde{T}^\varepsilon = 1$, then this is clear; let us suppose that $\tilde{T}^\varepsilon < 1$. This involves that

$$d\big(\overline{B}(h^\varepsilon(\tilde{T}^\varepsilon), \varepsilon R_0), \partial\Omega\big) = \overline{\delta}.$$

Using (8.10) we deduce

$$d(h^\varepsilon(\tilde{T}^\varepsilon), \partial\Omega) \leqslant 2\overline{\delta}.$$

With the triangle inequality and (8.9) we infer that

$$d\big(h^\varepsilon(\tilde{T}^\varepsilon), 0\big) \geqslant \frac{1}{2} d(0, \partial\Omega).$$

Now the relation (8.8) implies that for all $t \in [0, \tilde{T}^\varepsilon]$,

$$d\big(\mathcal{S}^\varepsilon(q^\varepsilon(t)), \partial\Omega\big) \geqslant d\big(\overline{B}(h^\varepsilon(t), \varepsilon R_0), \partial\Omega\big) \geqslant \overline{\delta}, \tag{8.13}$$

so that (8.12) is satisfied during $[0, \tilde{T}^\varepsilon]$. We deduce that $K\tilde{T}^\varepsilon \geqslant d(0,\partial\Omega)/2$, so $\tilde{T}^\varepsilon \geqslant \overline{T}$. Therefore for any $t \in [0, \overline{T}]$, for any $\varepsilon \in [0, \varepsilon_0]$, (8.13) holds true. This concludes the proof of Lemma 9. ☐

8.4. **Local passage to the limit: proof of Lemma 10.** We consider $\check{\delta} > 0$, $\check{T} > 0$ and $\varepsilon_1 > 0$ and suppose (3.27) to be satisfied. In particular we can apply Corollary 5 on the interval $[0, \check{T}]$ so that reducing $\varepsilon_1 > 0$ if necessary,

$$(|\ell^\varepsilon| + |\varepsilon\omega^\varepsilon|)_{\varepsilon \in (0,\varepsilon_1)} \text{ is bounded uniformly on } [0, \check{T}]. \tag{8.14}$$

Our goal is to pass to the limit in each term of (3.22) in Case (i) and in (3.23) in Case (ii).

8.4.1. *Case (i).* In that case, we work on (3.22).

First, thanks to (8.14) and (3.22), we deduce some uniform $W^{2,\infty}$ bounds on h^ε and $\varepsilon\vartheta^\varepsilon$. This involves that there exists a converging subsequence $(h^{\varepsilon_n}, \varepsilon_n \vartheta^{\varepsilon_n})$ of $(h^\varepsilon, \varepsilon\vartheta^\varepsilon)$:

$$(h^{\varepsilon_n}, \varepsilon_n \vartheta^{\varepsilon_n}) \longrightarrow (h^*, \Theta^*) \text{ in } W^{2,\infty} \text{ weak} - \star. \tag{8.15}$$

We now aim at characterizing the limit. The uniqueness of this limit will then prove the convergence as $\varepsilon \to 0^+$ (and not merely along a subsequence).

We start with noticing that the remaining term $\varepsilon H_r(\varepsilon, q^\varepsilon, \hat{p}^\varepsilon)$ converges to 0 strongly in L^∞. Hence we only have to pass to the limit in the terms $M_g^1(\hat{p}^\varepsilon)'$ and $F_{\cancel{\partial\Omega}, \vartheta^\varepsilon}(\tilde{p}^\varepsilon)$. For what concerns the term $M_g^1(\hat{p}^{\varepsilon_n})'$ we have

$$M_g^1(\hat{p}^{\varepsilon_n})' \longrightarrow M_g^1 \begin{pmatrix} (\Theta^*)'' \\ (h^*)'' \end{pmatrix} \text{ in } L^\infty \text{ weak} - \star.$$

Now consider

$$F_{\cancel{\partial\Omega}, \vartheta}(\tilde{p}^{\varepsilon_n}) = \gamma \begin{pmatrix} \zeta_{\vartheta^{\varepsilon_n}} \cdot \tilde{\ell}^{\varepsilon_n} \\ (\tilde{\ell}^{\varepsilon_n})^\perp - \varepsilon_n \omega^{\varepsilon_n} \zeta_{\vartheta^{\varepsilon_n}} \end{pmatrix},$$

see (2.14). On the one hand using $\omega^{\varepsilon_n} = (\vartheta^{\varepsilon_n})'$ and (2.15) we see that

$$\varepsilon_n \omega^{\varepsilon_n} \zeta_{\vartheta^{\varepsilon_n}} \rightharpoonup 0 \text{ in } W^{-1,\infty} \text{ weak} - \star. \tag{8.16}$$

On the other hand since the weak-$\star$ convergence in $W^{2,\infty}$ involves the strong $W^{1,\infty}$ one, we get that

$$\tilde{\ell}^{\varepsilon_n} \rightharpoonup (h^*)' - \gamma u_\Omega(h^*) \text{ in } W^{1,\infty} \text{ weak} - \star \text{ and } \big(h^*(0), (h^*)'(0)\big) = \big(0, \ell_0\big).$$

Hence we infer from the two last components of the system (3.22) that

$$m^1 (h^*)'' = \gamma[(h^*)' - \gamma u_c(h^*)]^\perp \text{ in } [0, \check{T}].$$

Due to the uniqueness of the solution of (1.22), this proves that $h^* = h_{(i)}$, and this concludes the proof of Lemma 10 for the part concerning the position of the center of mass.

We now turn to the part concerning the angle, that is the convergence of $\varepsilon\vartheta^\varepsilon$. We will use the following lemma (see [7]).

Lemma 31. *Let*

- $(\omega_n)_{n\in\mathbb{N}} \in W^{1,\infty}(0,T;\mathbb{R})^{\mathbb{N}}$, $(\varepsilon_n)_{n\in\mathbb{N}} \in (\mathbb{R}_*^+)^{\mathbb{N}}$ *with* $\varepsilon_n \to 0$ *as* $n \to +\infty$, *such that*

$$\varepsilon_n \omega_n \longrightarrow \overline{\rho} \quad in \quad W^{1,\infty}(0,T;\mathbb{R}) \ weak - \star \quad as \quad n \to +\infty; \tag{8.17}$$

- $(w_n)_{n\in\mathbb{N}} \in L^\infty(0,T;\mathbb{C})^{\mathbb{N}}$ *such that*

$$w_n \longrightarrow w \quad in \quad L^\infty(0,T;\mathbb{C}) \quad as \quad n \to +\infty; \tag{8.18}$$

- $\vartheta_n := \int_0^t \omega_n$.

Suppose that, on $(0,T)$,

$$\varepsilon_n \omega_n'(t) = \Re[w_n(t) \exp(-i\vartheta_n(t))]. \tag{8.19}$$

Then $\overline{\rho}$ *is constant on* $[0,T]$.

Above the notation $\Re$ stands for the real part. We consider the first component of the system (3.22): it allows to apply Lemma 31 to $\omega_n = \omega^{\varepsilon_n}$, $\vartheta_n = \vartheta^{\varepsilon_n}$ and

$$w_n := \frac{\gamma}{\mathcal{J}^1}\Big[(\zeta \cdot \ell^{\varepsilon_n}) - i(\zeta \cdot (\ell^{\varepsilon_n})^\perp)\Big] + \frac{\varepsilon_n}{\mathcal{J}^1} e^{i\vartheta^{\varepsilon_n}} H_r(\varepsilon_n, q^{\varepsilon_n}, \hat{p}^{\varepsilon_n}).$$

Using (8.15) and the initial data, we infer that $\Theta^* = 0$. This concludes the proof of Lemma 10 in Case (i).

8.4.2. *Case (ii).* Here we work with (3.23).

We first observe that in the left hand side ,

$$\big(\varepsilon^\alpha M_g^1 + \varepsilon^2 M_{a,\cancel{\partial\Omega},\vartheta^\varepsilon}\big)(\tilde{\tilde{p}}^\varepsilon)' \longrightarrow 0 \text{ in } W^{-1,\infty}.$$

Indeed, thanks to (8.14), $\varepsilon M_{a,\cancel{\partial\Omega},\vartheta^\varepsilon}$ is bounded in $W^{1,\infty}$ whereas $(\tilde{\tilde{p}}^\varepsilon)'$ is bounded in $W^{-1,\infty}$. On the other hand, M_g^1 is constant. Then the extra powers of ε in the left hand side allow to conclude.

Next, the term $\varepsilon\langle \Gamma_{\cancel{\partial\Omega},\vartheta^\varepsilon}, \tilde{\tilde{p}}^\varepsilon, \tilde{\tilde{p}}^\varepsilon\rangle$ converges to 0 in L^∞ since all terms in the brackets are bounded. In the same way, the terms $\varepsilon\gamma^2 \mathsf{E}_b^1(q^\varepsilon)$ and $\varepsilon^{\min(2,\alpha)} H_r(\varepsilon, q^\varepsilon, \tilde{\tilde{p}}^\varepsilon)$ converge strongly to 0 in L^∞.

Now let us consider the remaining terms in the two last lines of the equation (3.23). These are

$$(\tilde{\ell}^\varepsilon)^\perp - \hat{\omega}^\varepsilon \zeta_{\vartheta^\varepsilon}. \tag{8.20}$$

The last term converges weakly to 0 in $W^{-1,\infty}$ as seen in Case (i), see (8.16). Hence we infer that $\tilde{\ell}^{\varepsilon_n}$ converges weakly to 0 in $W^{-1,\infty}$. Due to the a priori estimate, this convergences occurs in L^∞ weak-$\star$. Again this is sufficient to deduce that the convergence of h^ε towards h^* is strong in L^∞, and

that $(h^*)' = \gamma u_c(h^*)$ and $h^*(0) = h_0$. The uniqueness of the solution of this Cauchy problem gives $h^* = h_{(ii)}$. Do to the uniqueness of the limit, the whole h^ε converges toward $h_{(ii)}$ as $\varepsilon \to 0^+$. This concludes the proof of Lemma 10 in Case (ii).

□

8.5. **Proof of Theorems 2 and 3.** We begin with Theorem 3, that is with Case (ii).

In that case, as mentioned below (1.28) it is well-known that the solution $h_{(ii)}$ is global in time, and in particular that there is no collision of the vortex point with the external boundary $\partial\Omega$. Hence let $T > 0$, and let us prove that for small $\varepsilon > 0$ the time of existence T^ε is larger than T and establish the convergences on the time interval $[0, T]$.

For such a T, we know that there exists $\overline{d} > 0$ such that

$$\forall t \in [0, T+1], \quad d(h_{(ii)}(t), \partial\Omega) \geqslant \overline{d}. \tag{8.21}$$

We let

$$\overline{T}^\varepsilon := \max \left\{ t > 0, \ d\big(B(h^\varepsilon(t), \varepsilon R_0), \partial\Omega\big) \geqslant \overline{d}/2 \right\}.$$

Let us recall that R_0 is defined in (8.7). Using Lemma 9 we deduce that, reducing $\overline{d}$ if necessary, we have that for some $\overline{\varepsilon} > 0$, $\inf_{\varepsilon \in (0, \overline{\varepsilon}]} \overline{T}^\varepsilon > 0$. Therefore

$$\tilde{T} := \liminf_{\varepsilon \to 0^+} \overline{T}^\varepsilon \quad \text{ satisfies } \quad \tilde{T} > 0.$$

Due to Corollary 5, there exists $K > 0$ and ε_0 such that for all $t \in [0, T+1]$ and $\varepsilon \in (0, \varepsilon_0)$, one has the following estimate

$$|\ell^\varepsilon| + |\varepsilon \omega^\varepsilon| \leqslant K \ \text{ as long as } q^\varepsilon(t) \text{ belongs to } Q^\varepsilon_{\overline{d}/2}. \tag{8.22}$$

Now we claim that

$$\tilde{T} \geqslant T + \frac{1}{2}. \tag{8.23}$$

Suppose that this is false. Then we have a sequence $\varepsilon_n \to 0^+$ such that $\overline{T}^{\varepsilon_n} \to \tilde{T} < T + \frac{1}{2}$. Now for any $\eta \in (0, \tilde{T})$, on the interval $[0, \tilde{T} - \eta]$, the condition $d(B(h^{\varepsilon_n}(t), \varepsilon R_0), \partial\Omega) \geqslant \overline{d}/2$ is satisfied for n large enough. Moreover, for such n, for all $t \in [0, \tilde{T} - \eta]$, (8.8) implies that

$$d(\mathcal{S}^{\varepsilon_n}(t), \partial\Omega) \geqslant d\big(\overline{B}(h^{\varepsilon_n}(t), \varepsilon_n R_0), \partial\Omega\big) \geqslant \overline{d}/2.$$

Hence applying Lemma 10, we deduce the uniform convergence of $(h^{\varepsilon_n})_n$ to $h_{(ii)}$ on $[0, \tilde{T} - \eta]$. In particular, as $n \to +\infty$, $d\big(h^{\varepsilon_n}(\tilde{T} - \eta), \partial\Omega\big) \to d\big(h_{(ii)}(\tilde{T} - \eta), \partial\Omega\big) \geqslant \overline{d}$, according to (8.21).

On the other hand by definition of $\overline{T}^{\varepsilon_n}$ we have $d\big(B(h^{\varepsilon_n}(\overline{T}^{\varepsilon_n}), \varepsilon_n R_0), \partial\Omega\big) = \overline{d}/2$. Using the triangle inequality and $\overline{T}^{\varepsilon_n} \to \tilde{T}$, we get a contradiction with (8.22). Hence (8.23) is valid, so that, reducing $\overline{\varepsilon}$ if necessary, we have $\inf_{\varepsilon \in (0, \overline{\varepsilon}]} \overline{T}^\varepsilon \geqslant T$.

Now, applying again (8.8) and Lemma 10, we reach the conclusion. This ends the proof of Theorem 3. □

The proof of Theorem 2 is exactly the same, except that the maximal time $T_{(i)}$ of existence of $h_{(i)}$ may be finite. Let $T \in (0, T_{(i)})$. It is then only a matter of replacing (8.21) and (8.23) respectively with

$$\forall t \in \left[0, \frac{T + T_{(i)}}{2}\right], \ \ d(h_{(i)}(t), \partial\Omega) \geqslant \overline{d}, \quad \text{ and with } \quad \tilde{T} := \liminf_{\varepsilon \to 0^+} \overline{T}^\varepsilon \geqslant T + \frac{T_{(i)} - T}{4},$$

to get the result in this case. This ends the proof of Theorem 2. □

9. Recasting of the system (1.1) as an ODE: Proof of Theorem 1

9.1. **Scheme of proof of Theorem 1.** Let us first give the scheme of the proof of Theorem 1.

The fluid velocity $u(q,\cdot)$ satisfies a div-curl type system in the doubly-connected domain $\mathcal{F}(q)$, constituted of (1.1b), (1.2), (1.1f), (1.1g), and of (1.3). When the solid position $q \in \mathcal{Q}$ and the right hand side s of these equations are given, the fluid velocity $u(q,\cdot)$ is determined in a unique way. Moreover, using (1.12), (1.7) and (1.8), the solution $u(q,\cdot)$ takes the form:

$$u(q,\cdot) = u_1(q,\cdot) + u_2(q,\cdot), \tag{9.1}$$

with

$$u_1(q,\cdot) := \nabla(\boldsymbol{\varphi}(q,\cdot)\cdot p) = \nabla\left(\sum_{j=1}^{3}\varphi_j(q,\cdot)p_j\right) \text{ and } u_2(q,\cdot) := \gamma\nabla^{\perp}\psi(q,\cdot), \tag{9.2}$$

where $q \in \mathcal{Q}$ and $p = (p_1,p_2,p_3) \in \mathbb{R}^3$. So besides the dependence with respect to $\mathcal{S}_0$, to Ω and to the space variable, u_1 depends on q and linearly on p while u_2 depends on q and linearly on γ.

In the system of equations (1.1), the initial data (1.1h) for the fluid is not required any longer (actually, it can be deduced from the given circulation γ and the initial data of the solid through the functions $\boldsymbol{\varphi}(0,\cdot)$ and $\psi(0,\cdot)$).

The pressure π can be recovered by means of Bernoulli's formula which is obtained by combining (1.1a) and (1.2), and which reads:

$$\nabla\pi = -\left(\frac{\partial u}{\partial t} + \frac{1}{2}\nabla|u^2|\right) \quad \text{in } \mathcal{F}(q). \tag{9.3}$$

For every q, p and γ, the pair (u,π) where u is given by (9.1) and π by (9.3) yields a solution to (1.1a-d). Equations (1.1g-h) can be summarized in the variational form:

$$m\ell'\cdot\ell^* + \mathcal{J}\omega'\omega^* = \int_{\partial\mathcal{S}(q)} \pi(\omega^*(x-h)^{\perp} + \ell^*)\cdot n\,\mathrm{d}s, \quad \forall\, p^* = (\omega^*,\ell^*) \in \mathbb{R}^3. \tag{9.4}$$

Let us associate with $(q,p^*) \in \mathcal{Q}\times\mathbb{R}^3$ the potential vector field

$$u^* := \nabla(\boldsymbol{\varphi}(q,\cdot)\cdot p^*), \tag{9.5}$$

which is defined on $\mathcal{F}(q)$. According to Bernoulli's formula (9.3) and upon an integration by parts, identity (9.4) can be turned into:

$$m\ell'\cdot\ell^* + \mathcal{J}\omega'\omega^* = -\int_{\mathcal{F}(q)}\left(\frac{\partial u}{\partial t} + \frac{1}{2}\nabla|u|^2\right)\cdot u^*\mathrm{d}x, \quad \forall\, p^* = (\omega^*,\ell^*) \in \mathbb{R}^3. \tag{9.6}$$

Therefore plugging the decomposition (9.1) into (9.6) leads to

$$\begin{aligned} m\ell'\cdot\ell^* + \mathcal{J}\omega'\omega^* + \int_{\mathcal{F}(q)}\left(\frac{\partial u_1}{\partial t} + \frac{1}{2}\nabla|u_1|^2\right)\cdot u^*\mathrm{d}x = &-\int_{\mathcal{F}(q)}\big(\frac{1}{2}\nabla|u_2|^2\big)\cdot u^*\mathrm{d}x \\ &-\int_{\mathcal{F}(q)}\big(\frac{\partial u_2}{\partial t} + \frac{1}{2}\nabla(u_1\cdot u_2)\big)\cdot u^*\mathrm{d}x, \end{aligned} \tag{9.7}$$

for all $p^* := (\omega^*,\ell^*) \in \mathbb{R}^3$.

Then the reformulation of Equations (1.1g-h) mentioned in Theorem 1 will follow from the three following lemmas which deal respectively with the left hand side of (9.7) and the two terms in the right hand side.

Lemma 32. *For any smooth curve $q(t)$ in $\mathcal{Q}$ and every $p^* = (\omega^*, \ell^*) \in \mathbb{R}^3$, the following identity holds:*

$$m\ell' \cdot \ell^* + \mathcal{J}\omega'\omega^* + \int_{\mathcal{F}(q)} \left(\frac{\partial u_1}{\partial t} + \frac{1}{2}\nabla |u_1|^2 \right) \cdot u^* \mathrm{d}x = M(q)p' \cdot p^* + \langle \Gamma(q), p, p\rangle \cdot p^*, \tag{9.8}$$

where $p := (\omega, \ell) = q'$, u^* is given by (9.5)*, u_1 is given by* (9.2)*, $M(q)$ and $\Gamma(q)$ are defined in* (1.9) *and* (1.11)*.*

Lemma 33. *For every $q \in \mathcal{Q}$ and every $p^* = (\omega^*, \ell^*) \in \mathbb{R}^3$, the following identity holds:*

$$- \int_{\mathcal{F}(q)} \left(\frac{1}{2}\nabla |u_2|^2 \right) \cdot u^* \mathrm{d}x = \gamma^2 E(q) \cdot p^*, \tag{9.9}$$

where u^* is given by (9.5)*, u_2 is given by* (9.2)*, $E(q)$ is defined in* (1.15)*.*

Lemma 34. *For any smooth curve $q(t)$ in $\mathcal{Q}$ and every $p^* = (\omega^*, \ell^*) \in \mathbb{R}^3$, the following identity holds:*

$$- \int_{\mathcal{F}(q)} \left(\frac{\partial u_2}{\partial t} + \nabla (u_1 \cdot u_2) \right) \cdot u^* \mathrm{d}x = \gamma \big(p \times B(q)\big) \cdot p^*, \tag{9.10}$$

where $p := (\omega, \ell) = q'$, u^* is given by (9.5)*, u_1 and u_2 are given by* (9.2)*, $B(q)$ is defined in* (1.15)*.*

Lemma 33 simply follows from an integration by parts. Let us consider Lemmas 32 and 34 as granted. Then gathering the results of Lemmas 32, 33 and 34 with (9.7), the conclusion of Theorem 1 follows. □

9.2. **Reformulation of the potential part: Proof of Lemma 32.** We start with observing that, under the assumptions of Lemma 32,

$$m\ell' \cdot \ell^* + \mathcal{J}\omega'\omega^* = M_g p' \cdot p^*. \tag{9.11}$$

Now in order to deal with the last term of the right hand side of (9.8) we use a Lagrangian strategy. For any q in $\mathcal{Q}$ and every p in $\mathbb{R}^3$, let us denote

$$\mathcal{E}_1(q,p) := \frac{1}{2} \int_{\mathcal{F}(q)} |u_1|^2 \mathrm{d}x, \tag{9.12}$$

where u_1 is given by (9.2). Thus $\mathcal{E}_1(q,p)$ denotes of the kinetic energy of the potential part u_1 of the flow associated with a body at position q with velocity p. It follows from classical shape derivative theory that $\mathcal{E}_1 \in C^\infty\big(\mathcal{Q} \times \mathbb{R}^3; [0,+\infty)\big)$. Below we make use of the first order partial derivatives that we now compute.

On the one hand the linearity of u_1 with respect to p and then an integration by parts leads to:

$$\frac{\partial \mathcal{E}_1}{\partial p} \cdot p^* = \int_{\mathcal{F}(q)} u_1 \cdot u^* \mathrm{d}x = \int_{\partial \mathcal{S}(q)} (\boldsymbol{\varphi} \cdot p)(u^* \cdot n) \, \mathrm{d}s.$$

Then, invoking Reynold's formula, we get:

$$\frac{\partial \mathcal{E}_1}{\partial p} \cdot p^* = \frac{\partial}{\partial q} \left(\int_{\mathcal{F}(q)} (\boldsymbol{\varphi} \cdot p) \, \mathrm{d}x \right) \cdot p^* - \int_{\mathcal{F}(q)} (\frac{\partial \boldsymbol{\varphi}}{\partial q} \cdot p) \cdot p^* \mathrm{d}x. \tag{9.13}$$

On the other hand, again using Reynold's formula, we have:

$$\frac{\partial \mathcal{E}_1}{\partial q} \cdot p^* = \int_{\mathcal{F}(q)} \left(\frac{\partial u_1}{\partial q} \cdot p^* \right) \cdot u_1 \, \mathrm{d}x + \frac{1}{2} \int_{\partial \mathcal{S}(q)} |u_1|^2 (u^* \cdot n) \, \mathrm{d}s. \tag{9.14}$$

Now the crucial quantity here is the Euler-Lagrange function:

$$\mathcal{EL} := \left(\frac{d}{dt}\frac{\partial \mathcal{E}_1}{\partial p} - \frac{\partial \mathcal{E}_1}{\partial q} \right) \cdot p^* . \tag{9.15}$$

Lemma 35. *For any smooth curve $q(t)$ in $\mathcal{Q}$, for every $p^* \in \mathbb{R}^3$, we have:*

$$\int_{\mathcal{F}(q)} \left(\frac{\partial u_1}{\partial t} + \frac{1}{2}\nabla |u_1|^2 \right) \cdot u^* \mathrm{d}x = \mathcal{EL} \tag{9.16}$$

where u_1 is given by (9.2)*, u^* is given by* (9.5) *and $\mathcal{EL}$ is given by* (9.15)*.*

Let us introduce a slight abuse of notations which simplifies the presentation of the proof of Lemma 35. For a smooth function $I(q,p)$, where (q,p) is running into $\mathcal{Q} \times \mathbb{R}^3$, and a smooth curve $q(t)$ in $\mathcal{Q}$ let us denote

$$\left(\frac{\partial}{\partial q}\frac{d}{dt} I(q,p) \right)(t) := (\frac{\partial}{\partial q} J)\big(q(t), q'(t), q''(t)\big),$$

where, for (q,p,r) in $\mathcal{Q} \times \mathbb{R}^3 \times \mathbb{R}^3$,

$$J(q,p,r) = p\frac{\partial I}{\partial q}(q,p) + r\frac{\partial I}{\partial p}(q,p). \tag{9.17}$$

Observe in particular that

$$\frac{d}{dt}\big(I(q(t), q'(t))\big) = J\big(q(t), q'(t), q''(t)\big),$$

and

$$\frac{d}{dt}\left(\frac{\partial I}{\partial q}(q(t), q'(t)) \right) = \left(\frac{\partial}{\partial q}\frac{d}{dt} I(q,p) \right)(t). \tag{9.18}$$

Below, in such circumstances, it will be comfortable to write

$$\frac{\partial}{\partial q}\left[J\big(q(t), q'(t), q''(t)\big) \right] \text{ instead of } \left(\frac{\partial J}{\partial q} \right)\big(q(t), q'(t), q''(t)\big),$$

and it will be understood that J is extended from $\big(q(t), q'(t), q''(t)\big)$ to general (q,p,r) by (9.17).

Proof of Lemma 35. We start with manipulating the right hand side of (9.16). Differentiating (9.13) with respect to t, we obtain:

$$\frac{d}{dt}\frac{\partial \mathcal{E}_1}{\partial p} \cdot p^* = \frac{d}{dt}\frac{\partial}{\partial q}\left(\int_{\mathcal{F}(q)} (\boldsymbol{\varphi} \cdot p)\, \mathrm{d}x \right) \cdot p^* - \frac{d}{dt}\left(\int_{\mathcal{F}(q)} (\frac{\partial \boldsymbol{\varphi}}{\partial q} \cdot p) \cdot p^* \mathrm{d}x \right). \tag{9.19}$$

With the abuse of notations mentioned above we commute the derivatives involved in the first term of the right hand side, so that the identity (9.19) can be rewritten as follows:

$$\frac{d}{dt}\frac{\partial \mathcal{E}_1}{\partial p} \cdot p^* = \frac{\partial}{\partial q}\frac{d}{dt}\left(\int_{\mathcal{F}(q)} (\boldsymbol{\varphi} \cdot p)\, \mathrm{d}x \right) \cdot p^* - \frac{d}{dt}\left(\int_{\mathcal{F}(q)} (\frac{\partial \boldsymbol{\varphi}}{\partial q} \cdot p) \cdot p^* \mathrm{d}x \right). \tag{9.20}$$

Moreover, using again Reynold's formula, we have:

$$\frac{d}{dt}\left(\int_{\mathcal{F}(q)} (\boldsymbol{\varphi} \cdot p)\, \mathrm{d}x \right) = \int_{\mathcal{F}(q)} \partial_t (\boldsymbol{\varphi} \cdot p)\, \mathrm{d}x + \int_{\partial \mathcal{S}(q)} (\boldsymbol{\varphi} \cdot p)(u_1 \cdot n)\, \mathrm{d}s \tag{9.21}$$

$$= \int_{\mathcal{F}(q)} \partial_t (\boldsymbol{\varphi} \cdot p)\, \mathrm{d}x + 2\mathcal{E}_1(q,p), \tag{9.22}$$

by integration by parts.

We infer from (9.20) and (9.21), again with the abuse of notations mentioned above, that:

$$\mathcal{EL} = \frac{\partial \mathcal{E}_1}{\partial q} + \frac{\partial}{\partial q}\left[\int_{\mathcal{F}(q)} \partial_t(\varphi \cdot p)\,\mathrm{d}x\right] \cdot p^* - \frac{d}{dt}\left(\int_{\mathcal{F}(q)} (\frac{\partial \varphi}{\partial q} \cdot p) \cdot p^* \mathrm{d}x\right). \tag{9.23}$$

Thanks to Reynold's formula, we get for the second term of the right hand side

$$\frac{\partial}{\partial q}\left[\int_{\mathcal{F}(q)} \partial_t(\varphi \cdot p)\,\mathrm{d}x\right] \cdot p^* = \int_{\mathcal{F}(q)} \frac{\partial}{\partial q}(\partial_t(\varphi \cdot p)) \cdot p^*\,\mathrm{d}x + \int_{\partial \mathcal{S}(q)} \partial_t(\varphi \cdot p)(u^* \cdot n)\,\mathrm{d}s, \tag{9.24}$$

and for the last one:

$$\frac{d}{dt}\left(\int_{\mathcal{F}(q)} \left(\frac{\partial \varphi}{\partial q} \cdot p\right) \cdot p^* \mathrm{d}x\right) = \int_{\mathcal{F}(q)} \partial_t\left(\left(\frac{\partial \varphi}{\partial q} \cdot p\right) \cdot p^*\right)\mathrm{d}x + \int_{\partial \mathcal{S}(q)} \left(\left(\frac{\partial \varphi}{\partial q} \cdot p\right) \cdot p^*\right)(u_1 \cdot n)\,\mathrm{d}s.$$

Using again (9.18) for the first term and integrating by parts the second one, we obtain:

$$\frac{d}{dt}\left(\int_{\mathcal{F}(q)} \left(\frac{\partial \varphi}{\partial q} \cdot p\right) \cdot p^* \mathrm{d}x\right) = \int_{\mathcal{F}(q)} \frac{\partial}{\partial q}(\partial_t(\varphi \cdot p)) \cdot p^*\,\mathrm{d}x + \int_{\mathcal{F}(q)} \left(\frac{\partial u_1}{\partial q} \cdot p^*\right) \cdot u_1\,\mathrm{d}x. \tag{9.25}$$

Plugging the expressions (9.14), (9.24) and (9.25) into (9.23) and simplifying, we end up with:

$$\mathcal{EL} = \int_{\partial \mathcal{S}(q)} \left[\partial_t(\varphi \cdot p) + \frac{1}{2}|u_1|^2\right](u^* \cdot n)\,\mathrm{d}s.$$

Upon an integration by parts, we recover (9.16) and the proof is then completed. □

Now, we observe that the kinetic energy $\mathcal{E}_1(q,p)$ of the potential part of the flow, as defined by (9.12), can be rewritten as:

$$\mathcal{E}_1(q,p) = \frac{1}{2} M_a(q) p \cdot p, \tag{9.26}$$

where $M_a(q)$ is defined by (1.9). Indeed this allows us to prove the following result.

Lemma 36. *For any smooth curve $q(t)$ in $\mathcal{Q}$, for every $p^* \in \mathbb{R}^3$, we have:*

$$\mathcal{EL} = M_a(q) p' \cdot p^* + \langle \Gamma(q), p, p \rangle \cdot p^*, \tag{9.27}$$

with $p := q'$, $\mathcal{EL}$ is given by (9.15), $M_a(q)$ defined by (1.9) and $\Gamma(q)$ associated with $M(q)$ by the Christoffel formula (1.11a)-(1.11b).

Proof of Lemma 36. Using (9.26) in the definition (9.15) of $\mathcal{EL}$ we have

$$\mathcal{EL} = M_a(q) p' \cdot p^* + \Big(\big(DM_a(q) \cdot p\big)p\Big) \cdot p^* - \frac{1}{2}\Big(\big(DM_a(q) \cdot p^*\big)p\Big) \cdot p.$$

Let us recall the notation $(M_a)^k_{i,j}(q)$ in (1.11c) and let the notation $\sum$ stands for $\sum_{1 \leqslant i,j,k \leqslant 3}$ for the rest of this proof. Then

$$\mathcal{EL} = M_a\, p' \cdot p^* + \sum (M_a)^k_{i,j}\, p_k p_j p^*_i - \frac{1}{2}\sum (M_a)^k_{i,j}\, p_i p_j p^*_k.$$

A symmetrization of the second term of the right hand side above leads to

$$\mathcal{EL} = M_a\, p' \cdot p^* + \frac{1}{2}\Big(\sum (M_a)^k_{i,j}\, p_k p_j p^*_i + \sum (M_a)^j_{i,k}\, p_k p_j p^*_i - \sum (M_a)^k_{i,j}\, p_i p_j p^*_k\Big),$$

and then to the result by exchanging i and k in the last sum. □

Then Lemma 32 straightforwardly results from the combination of (9.11), Lemmas 35 and 36. □

9.3. **Reformulation of the cross part: Proof of Lemma 34.** Assume that $q := (\vartheta, h)$ and $p := (\omega, \ell) = q'$ and recall that

$$u_2 := \gamma \nabla^\perp \psi(q, \cdot), \quad u_1 := \nabla(\boldsymbol{\varphi}(q, \cdot) \cdot p) \text{ and } u^* := \nabla(\boldsymbol{\varphi}(q, \cdot) \cdot p^*).$$

We first observe that

$$\int_{\mathcal{F}(q)} \left(\frac{\partial u_2}{\partial t} \right) \cdot u^* \mathrm{d}x = -\gamma \int_{\partial \mathcal{S}(q)} \left(\frac{\partial}{\partial t} \big(\psi(q, \cdot) \big) \right) \big(\frac{\partial \boldsymbol{\varphi}}{\partial \tau}(q, \cdot) \cdot p^* \big) \mathrm{d}s. \tag{9.28}$$

Let us emphasize that there is no contribution on $\partial\Omega$ since $\psi(q, \cdot)$ is vanishing there.

Now we have the following result.

Lemma 37. *On $\partial \mathcal{S}(q)$, we have*

$$\frac{\partial}{\partial t} \big(\psi(q, \cdot) \big) = -\frac{\partial \psi}{\partial n}(q, \cdot) \left(\frac{\partial \boldsymbol{\varphi}}{\partial n}(q, \cdot) \cdot p \right) + DC(q) \cdot p. \tag{9.29}$$

Proof of Lemma 37. We start with the observation that

$$\frac{\partial}{\partial t} \big(\psi(q, \cdot) \big) = \frac{\partial \psi}{\partial q}(q, \cdot) \cdot p, \tag{9.30}$$

is the derivative of the function $\psi(q, \cdot)$ when the boundary $\partial \mathcal{S}(q)$ undergoes a rigid displacement of velocity $w = \omega(x - h)^\perp + \ell$.

Then we differentiate the identity:

$$\psi(q, R(\vartheta)X + h) = C(q), \qquad \text{for } X \in \partial \mathcal{S}_0,$$

with respect to q in the direction p. We obtain:

$$\frac{\partial \psi}{\partial q}(q, x) \cdot p + \nabla \psi(q, x) \cdot w = DC(q) \cdot p, \qquad \text{for } x \in \partial \mathcal{S}(q). \tag{9.31}$$

Since $\psi(q, \cdot)$ is constant on $\partial \mathcal{S}(q)$, its tangential derivative is zero. Besides, on $\partial \mathcal{S}(q)$ we have $w \cdot n = u_1 \cdot n = \frac{\partial \boldsymbol{\varphi}}{\partial n}(q, \cdot) \cdot p$ and we then get

$$\nabla \psi(q, x) \cdot w = \frac{\partial \psi}{\partial n}(q, x) \big(\frac{\partial \boldsymbol{\varphi}}{\partial n}(q, x) \cdot p \big) \text{ for } x \in \partial \mathcal{S}(q). \tag{9.32}$$

Gathering (9.30), (9.31) and (9.32) we obtain (9.29). □

Plugging now (9.29) into (9.28) we deduce that:

$$\int_{\mathcal{F}(q)} \left(\frac{\partial u_2}{\partial t} \right) \cdot u^* \mathrm{d}x = \gamma \int_{\partial \mathcal{S}(q)} \frac{\partial \psi}{\partial n} \left(\frac{\partial \boldsymbol{\varphi}}{\partial n} \cdot p \right) \left(\frac{\partial \boldsymbol{\varphi}}{\partial \tau} \cdot p^* \right) \mathrm{d}s. \tag{9.33}$$

On the other hand, integrating by parts and using that $u_2 = -\gamma \frac{\partial \psi}{\partial n} \tau$ on $\partial \mathcal{S}(q)$, we get:

$$\int_{\mathcal{F}(q)} \nabla(u_1 \cdot u_2) \cdot u^* \mathrm{d}x = \gamma \int_{\partial \mathcal{S}(q)} (u_1 \cdot u_2)(u^* \cdot n) \mathrm{d}s. \tag{9.34}$$

Adding (9.33) and (9.34), we get:

$$\int_{\mathcal{F}(q)} \left(\frac{\partial u_2}{\partial t} + \nabla(u_1 \cdot u_2) \right) \cdot u^* \mathrm{d}x = \gamma \int_{\partial \mathcal{S}(q)} \frac{\partial \psi}{\partial n} \left[\left(\frac{\partial \boldsymbol{\varphi}}{\partial n} \cdot p \right) \left(\frac{\partial \boldsymbol{\varphi}}{\partial \tau} \cdot p^* \right) - \left(\frac{\partial \boldsymbol{\varphi}}{\partial n} \cdot p^* \right) \left(\frac{\partial \boldsymbol{\varphi}}{\partial \tau} \cdot p \right) \right] \mathrm{d}s,$$

which is (9.10). □

9.4. **Proof of Proposition 2.** This section is devoted to the proof of Proposition 2. We will use the matrix $\widetilde{M}_a(q)$ given by

$$\widetilde{M}_a(q) := \mathcal{R}(\vartheta)^t\, M_a(q)\mathcal{R}(\vartheta), \tag{9.35}$$

where we recall that $\mathcal{R}(\vartheta)$ is defined by (2.12). We also introduce the real valued function of $q = (\vartheta, h) \in \mathcal{Q}$, $p \in \mathbb{R}^3$ and $p^* \in \mathbb{R}^3$:

$$\Xi_1(q,p,p^*) := \left[\left(\frac{\partial \widetilde{M}_a}{\partial q}(q)\cdot p^*\right)\mathcal{R}(\vartheta)^t\, p\right]\cdot \mathcal{R}(\vartheta)^t\, p,$$
$$\Xi_3(q,p,p^*) := \left[\left(\frac{\partial \widetilde{M}_a}{\partial q}(q)\cdot p\right)\mathcal{R}(\vartheta)^t\, p\right]\cdot \mathcal{R}(\vartheta)^t\, p^*.$$

Let us emphasize that the indexes above are chosen in order to recall the position where p^* appears in the three occurrences of p and p^* of $\Xi_1(q,p,p^*)$ and $\Xi_3(q,p,p^*)$.

Similarly we define, for $p = (\omega, \ell)$, $p^* = (\omega^*, \ell^*)$,

$$\Upsilon_1(q,p,p^*) := \omega^* M_a(q)p\cdot\begin{pmatrix}0\\ \ell^\perp\end{pmatrix},\quad \Upsilon_2(q,p,p^*) := \omega M_a(q)p^*\cdot\begin{pmatrix}0\\ \ell^\perp\end{pmatrix},$$
$$\text{and } \Upsilon_3(q,p,p^*) := \omega M_a(q)p\cdot\begin{pmatrix}0\\ \ell^{*\perp}\end{pmatrix}.$$

Proposition 2 straightforwardly results from the following three lemmas.

Lemma 38. *For any $(q,p,p^*) \in \mathcal{Q}\times\mathbb{R}^3\times\mathbb{R}^3$, we have:*

$$\langle\Gamma(q),p,p\rangle\cdot p^* = \Upsilon_1(q,p,p^*) - \Upsilon_2(q,p,p^*) - \Upsilon_3(q,p,p^*) + \Xi_3(q,p,p^*) - \frac{1}{2}\Xi_1(q,p,p^*). \tag{9.36}$$

Lemma 39. *For any $(q,p,p^*) \in \mathcal{Q}\times\mathbb{R}^3\times\mathbb{R}^3$, we have:*

$$\Upsilon_1(q,p,p^*) - \Upsilon_2(q,p,p^*) - \Upsilon_3(q,p,p^*) = \langle\Gamma_{\mathcal{S}}(q),p,p\rangle\cdot p^*,$$

where $\Gamma_{\mathcal{S}}(q)$ is defined in (1.11).

Lemma 40. *For any $(q,p,p^*) \in \mathcal{Q}\times\mathbb{R}^3\times\mathbb{R}^3$, we have:*

$$\Xi_3(q,p,p^*) - \frac{1}{2}\Xi_1(q,p,p^*) = \langle\Gamma_{\partial\Omega}(q),p,p\rangle\cdot p^*, \tag{9.37}$$

where $\Gamma_{\partial\Omega}(q)$ is defined in (1.11).

□

Before proving these three lemmas, let us introduce a few notations. For every $q = (\vartheta, h) \in \mathcal{Q}$, we introduce the change of variables $y = R(\vartheta)^t(x-h)$, the domains

$$\widetilde{\Omega}(q) := R(\vartheta)^t(\Omega - h),\quad \widetilde{\mathcal{F}}(q) := R(\vartheta)^t(\mathcal{F}(q)-h) = \widetilde{\Omega}(q)\setminus\bar{\mathcal{S}}_0,$$

and the functions $\widetilde{\varphi}_i(q,y)$ such that, denoting $\widetilde{\boldsymbol{\varphi}}(q,\cdot) := (\widetilde{\varphi}_1(q,\cdot), \widetilde{\varphi}_2(q,\cdot), \widetilde{\varphi}_3(q,\cdot))$, we have

$$\widetilde{\boldsymbol{\varphi}}(q,y) := \mathcal{R}(\vartheta)^t\boldsymbol{\varphi}(q,x),\qquad y\in\widetilde{\mathcal{F}}(q).$$

For every $j = 1,2,3$, the functions $\widetilde{\varphi}_j(q,\cdot)$ are harmonic in $\widetilde{\mathcal{F}}(q)$ and satisfy:

$$\frac{\partial\widetilde{\varphi}_j}{\partial n}(q,y) = \begin{cases} y^\perp\cdot n & j=1;\\ n_{j-1} & j=2,3\end{cases}\qquad \text{on } \partial\mathcal{S}_0, \tag{9.38a}$$

$$\frac{\partial\widetilde{\varphi}_j}{\partial n}(q,y) = 0\quad (j=1,2,3)\qquad \text{on } \partial\widetilde{\Omega}(q). \tag{9.38b}$$

Therefore the matrix $\widetilde{M}_a(q)$ defined in (9.35) can be recast as

$$\widetilde{M}_a(q) = \int_{\partial \mathcal{S}_0} \widetilde{\boldsymbol{\varphi}}(q) \otimes \frac{\partial \widetilde{\boldsymbol{\varphi}}}{\partial n}(q)\mathrm{d}s. \tag{9.39}$$

Proof of Lemma 38. Let $q(t)$ be a smooth curve in $\mathcal{Q}$, defined in a neighborhood of 0 such that $q(0) = q$ and $q'(0) = p$. For every $p^* = (\omega^*, \ell^*) \in \mathbb{R}^3$, using the expression of $\mathcal{E}_1(q,p)$ given by (9.26) lead to, on the one hand:

$$\begin{aligned} \frac{d}{dt}\frac{\partial \mathcal{E}_1}{\partial p} \cdot p^* &= \frac{d}{dt}\left(\widetilde{M}_a(q)\mathcal{R}(\vartheta)^t\, p\right) \cdot \mathcal{R}(\vartheta)^t\, p^* - \Upsilon_3(q,p,p^*) \\ &= M_a(q)p' \cdot p^* - \Upsilon_2(q,p,p^*) + \Xi_3(q,p,p^*) - \Upsilon_3(q,p,p^*). \end{aligned} \tag{9.40}$$

On the other hand, we get:

$$\frac{\partial \mathcal{E}_1}{\partial q} \cdot p^* = -\Upsilon_1(q,p,p^*) + \frac{1}{2}\Xi_1(q,p,p^*). \tag{9.41}$$

Gathering (9.40) and (9.41) and (9.27) we deduce (9.36). □

Proof of Lemma 39. On the one hand, invoking the symmetry of $M(q)$, we get:

$$\Upsilon_2(q,p,p^*) = \omega M_a(q)\begin{pmatrix} 0 \\ \ell^\perp \end{pmatrix} \cdot p^*. \tag{9.42}$$

On the other hand:

$$-\Upsilon_3(q,p,p^*) + \Upsilon_1(q,p,p^*) = \begin{pmatrix} -\mathbf{P}_a^\perp \cdot \ell \\ \omega \mathbf{P}_a^\perp \end{pmatrix} \cdot p^* = -\left[\begin{pmatrix} 0 \\ P_a \end{pmatrix} \times p\right] \cdot p^*. \tag{9.43}$$

Now gathering (9.42) and (9.43) concludes the proof of Lemma 39. □

Proof of Lemma 40. We first prove the following lemma.

Lemma 41. *For $i,j = 1,2,3$, for every $\hat{q} = (\hat{\vartheta}, \hat{h}) \in \mathcal{Q}$ and every $p^* = (\omega^*, \ell^*) \in \mathbb{R}^3$, we have:*

$$\frac{\partial}{\partial q}\left(\int_{\widetilde{\mathcal{F}}(q)} \nabla \widetilde{\varphi}_i(q) \cdot \nabla \widetilde{\varphi}_j(q)\mathrm{d}y\right)\Bigg|_{q=\hat{q}} \cdot p^* = -\int_{\partial \widetilde{\Omega}(\hat{q})} \frac{\partial \widetilde{\varphi}_i}{\partial \tau}\frac{\partial \widetilde{\varphi}_j}{\partial \tau}(w^* \cdot n)\mathrm{d}s, \tag{9.44}$$

with $w^*(\hat{q}, p^*, \cdot) := -\omega^* \cdot^\perp - R(\hat{\vartheta})^t\, \ell^*$.

Let us take Lemma 41 for granted for a while and let us see how to conclude the proof of Lemma 40. Applying now the change of variables $x = R(\hat{\vartheta})y + \hat{h}$, we deduce that:

$$\begin{aligned} \frac{\partial \widetilde{M}_a}{\partial q}(\hat{q}) \cdot p^* &= \left(\frac{\partial}{\partial q}\left(\int_{\widetilde{\mathcal{F}}(q)} \nabla \widetilde{\varphi}_i(q) \cdot \nabla \widetilde{\varphi}_j(q)\mathrm{d}y\right)\Bigg|_{q=\hat{q}} \cdot p^*\right)_{1 \leqslant i,j \leqslant 3} \\ &= -\left(\int_{\partial \widetilde{\Omega}(\hat{q})} \frac{\partial \widetilde{\varphi}_i}{\partial \tau}\frac{\partial \widetilde{\varphi}_j}{\partial \tau}(w^* \cdot n)\mathrm{d}s\right)_{1 \leqslant i,j \leqslant 3} \\ &= \mathcal{R}(\hat{\vartheta})^t \left(\int_{\partial \Omega} \frac{\partial \varphi_i}{\partial \tau}(\hat{q})\frac{\partial \varphi_j}{\partial \tau}(\hat{q})(\tilde{w}^* \cdot n)\mathrm{d}s\right)_{1 \leqslant i,j \leqslant 3} \mathcal{R}(\hat{\vartheta}), \end{aligned}$$

with $\tilde{w}^* := \omega^*(x-h)^\perp + \ell^*$.

Therefore, applying this with $(\hat{q}, p^*) = (q, p^*)$ and with $(\hat{q}, p^*) = (q, p)$, we get

$$\Xi_3(q,p,p^*) - \frac{1}{2}\Xi_1(q,p,p^*) = \sum [\Lambda^l_{ij}(q)p_l p_j - \frac{1}{2}\Lambda^i_{jl}(q)p_l p_j]p^*_i,$$

where the notation $\sum$ stands for $\sum_{1\leqslant i,j,l\leqslant 3}$ for the rest of this proof and for every $k=1,2,3$ the matrices $\chi^k(\hat{q})$ are given by:

$$\Lambda^k(\hat{q})=\left(\int_{\partial\Omega}\frac{\partial\varphi_i}{\partial\tau}(\hat{q})\frac{\partial\varphi_j}{\partial\tau}(\hat{q})K_k(\hat{q},\cdot)\mathrm{d}s\right)_{1\leqslant i,j\leqslant 3},$$

where we recall that $K_1(\hat{q},\cdot)=(x-\hat{h})^\perp\cdot n$ and $K_j(\hat{q},\cdot)=n_{j-1}$ $(j=2,3)$ on $\partial\Omega$.

The quadratic form in p can be symmetrized as follows:

$$\sum\left[\Lambda^l_{ij}(q)p_lp_j-\frac{1}{2}\Lambda^i_{jl}(q)p_lp_j\right]p_i^*=\frac{1}{2}\sum\left[\Lambda^l_{ij}+\Lambda^j_{il}-\Lambda^i_{jl}\right](q)\,p_l\,p_j\,p_i^*,$$

which leads to the equality (9.37) in the statement of Lemma 40.

∎

Now we give the proof of Lemma 41.

Proof of Lemma 41. The quantity

$$\frac{\partial}{\partial q}\left(\int_{\widetilde{\mathcal{F}}(q)}\nabla\widetilde{\varphi}_i(q)\cdot\nabla\widetilde{\varphi}_j(q)\mathrm{d}y\right)\bigg|_{q=\hat{q}}\cdot p^* \tag{9.45}$$

can be interpreted as the time derivative of the quantity between parentheses, when the fluid outer boundary $\partial\widetilde{\Omega}(\hat{q})$ undergoes a rigid displacement of velocity w^*.

More precisely, denote by χ a cut-off function, compactly supported, valued in $[0,1]$ and such that $\chi=1$ in a neighborhood of $\partial\widetilde{\Omega}(\hat{q})$ and $\chi=0$ in a neighborhood of $\mathcal{S}_0$. Then, denote by $\xi(t,\cdot)$ the flow associated with the ODE:

$$\xi'(t,y)=\chi(\xi(t,y))w^*(t,\xi(t,y)),\qquad\text{for } t>0,\ \text{with } \xi(0,y)=y.$$

Notice that:

$$\xi(t,y)=R(-t\omega^*)y-tR(\hat{\vartheta})^t\,\ell^*,$$

in a neighborhood of $\partial\widetilde{\Omega}(\hat{q})$ and $\xi(t,y)=y$ in a neighborhood of $\partial\mathcal{S}_0$.

For every t small, define

$$\Omega_t:=\xi(t,\widetilde{\Omega}(\hat{q}))\text{ and }\mathcal{F}_t:=\xi(t,\widetilde{\mathcal{F}}(\hat{q})).$$

For $j=1,2,3$, let $\widetilde{\varphi}^t_j$ be harmonic in $\mathcal{F}_t$ and satisfy the Neumann boundary conditions:

$$\frac{\partial\widetilde{\varphi}^t_j}{\partial n}=\begin{cases}(y-\hat{h})^\perp\cdot n & j=1;\\ n_{j-1} & j=2,3\end{cases}\qquad\text{on } \partial\mathcal{S}_0, \tag{9.46a}$$

$$\frac{\partial\widetilde{\varphi}^t_j}{\partial n}=0\quad(j=1,2,3)\qquad\text{on } \partial\Omega_t. \tag{9.46b}$$

With these settings, the quantity (9.45) can be rewritten as:

$$\frac{d}{dt}\left(\int_{\mathcal{F}_t}\nabla\widetilde{\varphi}^t_i\cdot\nabla\widetilde{\varphi}^t_j\,\mathrm{d}x\right)\bigg|_{t=0}. \tag{9.47}$$

According to Reynold's formula, it can be expended as follows:

(9.48)
$$\frac{d}{dt}\left(\int_{\mathcal{F}_t}\nabla\widetilde{\varphi}^t_i\cdot\nabla\widetilde{\varphi}^t_j\,\mathrm{d}x\right)\bigg|_{t=0}=\int_{\mathcal{F}_{t=0}}\nabla\widetilde{\varphi}'_i\cdot\nabla\widetilde{\varphi}_j\,\mathrm{d}x+\int_{\mathcal{F}_{t=0}}\nabla\widetilde{\varphi}_i\cdot\nabla\widetilde{\varphi}'_j\,\mathrm{d}x+\int_{\partial\Omega_{t=0}}(\nabla\widetilde{\varphi}_i\cdot\nabla\widetilde{\varphi}_j)(w^*\cdot n)\mathrm{d}s,$$

where

$$\widetilde{\varphi}_j' := \frac{\partial \widetilde{\varphi}_j^t}{\partial t}|_{t=0}.$$

Lemma 42. *For $j = 1, 2, 3$, the function $\widetilde{\varphi}_j'$ is harmonic in $\mathcal{F}_{t=0} = \widetilde{\mathcal{F}}(\hat{q})$, satisfies*

$$\frac{\partial \widetilde{\varphi}_j'}{\partial n} = 0 \text{ on } \partial \mathcal{S}_0 \tag{9.49}$$

and

$$\frac{\partial \widetilde{\varphi}_j'}{\partial n} = \frac{\partial}{\partial \tau}\Big((w^* \cdot n)\frac{\partial \widetilde{\varphi}_j}{\partial \tau}\Big) \text{ on } \partial \Omega_{t=0} = \partial \widetilde{\Omega}(\hat{q}). \tag{9.50}$$

Once Lemma 42 is proved, (9.44) follows from (9.48) and an integration by parts. □

Proof of Lemma 42. The function $\widetilde{\varphi}_j'$ is defined and harmonic in $\mathcal{F}_{t=0} = \widetilde{\mathcal{F}}(\hat{q})$ and the boundary conditions are obtained by differentiating with respect to t, at $t = 0$, the identities on the fixed boundaries $\partial \mathcal{S}_0$ and $\partial \widetilde{\Omega}(\hat{q})$:

$$\frac{\partial \widetilde{\varphi}_j^t}{\partial n}\big(\xi(t,\cdot)\big) = \begin{cases} (y - \hat{h})^\perp \cdot n & j = 1; \\ n_{j-1} & j = 2, 3; \end{cases} \quad \text{on } \partial \mathcal{S}_0, \tag{9.51a}$$

$$\frac{\partial \widetilde{\varphi}_j^t}{\partial n}\big(\xi(t,\cdot)\big) = 0 \quad (j = 1, 2, 3) \qquad \text{on } \partial \widetilde{\Omega}(\hat{q}). \tag{9.51b}$$

Let us focus on the proof of (9.50), the proof of (42) being quite similar with some simplifications.

On $\partial \widetilde{\Omega}(\hat{q})$ we can write that:

$$\frac{d}{dt}\left(\frac{\partial \widetilde{\varphi}_j^t}{\partial n}\big(\xi(t,\cdot)\big)\right)|_{t=0} = \frac{\partial \widetilde{\varphi}_j'}{\partial n} + \langle D^2 \widetilde{\varphi}_j, w^*, n\rangle + \omega^* \frac{\partial \widetilde{\varphi}_j}{\partial \tau}, \tag{9.52}$$

where the last term is obtained by remarking that $n(\xi(t,\cdot)) = R(-t\omega^*)n$.

Therefore by taking the derivative at $t = 0$ of the identity (9.51b) and using (9.52) we obtain

$$\begin{aligned} \frac{\partial \widetilde{\varphi}_j'}{\partial n} &= -\langle D^2 \widetilde{\varphi}_j, w^*, n\rangle - \omega^* \frac{\partial \widetilde{\varphi}_j}{\partial \tau} \\ &= -\frac{\partial^2 \widetilde{\varphi}_j}{\partial n^2}(w^* \cdot n) - \langle D^2 \widetilde{\varphi}_j, \tau, n\rangle (w^* \cdot \tau) - \omega^* \frac{\partial \widetilde{\varphi}_j}{\partial \tau}, \end{aligned} \tag{9.53}$$

by decomposing w^* into normal and tangential parts.

Taking the tangential derivative of this identity (9.38b), we get:

$$\langle D^2 \widetilde{\varphi}_j, \tau, n\rangle + \kappa \frac{\partial \widetilde{\varphi}^j}{\partial \tau} = 0, \tag{9.54}$$

where we used to relation $\frac{\partial n}{\partial \tau} = \kappa \tau$ with κ the local curvature of $\partial \widetilde{\Omega}(\hat{q})$. Plugging (9.54) into (9.53) yields the identity:

$$\begin{aligned} \frac{\partial \widetilde{\varphi}_j'}{\partial n} &= -\frac{\partial^2 \widetilde{\varphi}_j}{\partial n^2}(w^* \cdot n) + \Big(\kappa (w^* \cdot \tau) - \omega^*\Big)\frac{\partial \widetilde{\varphi}_j}{\partial \tau} \\ &= -\frac{\partial^2 \widetilde{\varphi}_j}{\partial n^2}(w^* \cdot n) + \big(\frac{\partial}{\partial \tau}(w^* \cdot n)\big)\frac{\partial \widetilde{\varphi}_j}{\partial \tau}. \end{aligned} \tag{9.55}$$

On $\partial \widetilde{\Omega}(\hat{q})$, we have with local coordinates:

$$\Delta \widetilde{\varphi}_j = \frac{\partial^2 \widetilde{\varphi}_j}{\partial \tau^2} - \kappa \frac{\partial \widetilde{\varphi}_j}{\partial n} + \frac{\partial^2 \widetilde{\varphi}_j}{\partial n^2}.$$

Since $\widetilde{\varphi}_j$ is harmonic and $\frac{\partial \widetilde{\varphi}_j}{\partial n} = 0$ on $\partial\widetilde{\Omega}(\hat{q})$, we deduce that $\frac{\partial^2 \widetilde{\varphi}_j}{\partial n^2} = -\frac{\partial^2 \widetilde{\varphi}_j}{\partial \tau^2}$ on $\partial\widetilde{\Omega}(\hat{q})$, and therefore

$$\frac{\partial \widetilde{\varphi}_j'}{\partial n} = (w^* \cdot n)\frac{\partial^2 \widetilde{\varphi}_j}{\partial \tau^2} + \big(\frac{\partial}{\partial \tau}(w^* \cdot n)\big)\frac{\partial \widetilde{\varphi}_j}{\partial \tau},$$

which is (9.50). □

9.5. **Proof of** (2.22)**.** By definition we have

$$C(q) = -\int_{\mathcal{F}(q)} \nabla\psi(q,\cdot)\cdot\nabla\psi(q,\cdot)\,\mathrm{d}x.$$

Thus, by Reynold's formula we infer that

$$DC(q)\cdot p = 2\int_{\mathcal{F}(q)} \nabla\left(\frac{\partial \psi}{\partial q}\cdot p\right)\cdot\nabla\psi\,\mathrm{d}x + \int_{\partial\mathcal{S}(q)} |\nabla\psi|^2 u_1\cdot n\,\mathrm{d}s. \tag{9.56}$$

Using integration by parts, we get

$$\int_{\mathcal{F}(q)} \nabla\left(\frac{\partial \psi}{\partial q}\cdot p\right)\cdot\nabla\psi\,\mathrm{d}x = \int_{\partial\mathcal{S}(q)} \big(\frac{\partial \psi}{\partial q}\cdot p\big)\frac{\partial \psi}{\partial n}\,\mathrm{d}s. \tag{9.57}$$

Gathering (9.31) and (9.32) yields

$$\frac{\partial \psi}{\partial q}(q,x)\cdot p = DC(q)\cdot p - \frac{\partial \psi}{\partial n}(q,x)\frac{\partial \boldsymbol{\varphi}}{\partial n}(q,x)\cdot p. \qquad \text{for } x\in\partial\mathcal{S}(q), \tag{9.58}$$

Combining (9.57) and (9.57) we obtain

$$\begin{aligned}\int_{\mathcal{F}(q)} \nabla\left(\frac{\partial \psi}{\partial q}\cdot p\right)\cdot\nabla\psi\,\mathrm{d}x &= \int_{\partial\mathcal{S}(q)} (DC(q)\cdot p)\frac{\partial \psi}{\partial n}\,\mathrm{d}s - \int_{\partial\mathcal{S}(q)} |\frac{\partial \psi}{\partial n}|^2\frac{\partial \boldsymbol{\varphi}}{\partial n}\cdot p\,\mathrm{d}s\\ &= -DC(q)\cdot p - \int_{\partial\mathcal{S}(q)} |\frac{\partial \psi}{\partial n}|^2\frac{\partial \boldsymbol{\varphi}}{\partial n}\cdot p\,\mathrm{d}s,\end{aligned} \tag{9.59}$$

thanks to (1.12d).

On the other hand since $\psi(q,\cdot)$ is constant on $\partial\mathcal{S}(q)$, we get

$$\int_{\partial\mathcal{S}(q)} |\nabla\psi|^2 u_1\cdot n\,\mathrm{d}s = \int_{\partial\mathcal{S}(q)} |\frac{\partial \psi}{\partial n}|^2\frac{\partial \boldsymbol{\varphi}}{\partial n}\cdot p\,\mathrm{d}s. \tag{9.60}$$

Gathering (9.56), (9.59), (9.60) and (1.15b) leads to the result. □

9.6. **Energy conservation: Two proofs of Proposition 1.**

9.6.1. *First proof: with the PDE formulation.* In the PDE framework, that is (1.1), we can introduce the total kinetic energy $\hat{\mathcal{E}}$ of the system "fluid+solid" by

$$\hat{\mathcal{E}} := \frac{1}{2}\int_{\mathcal{F}(q)} u^2\,\mathrm{d}x + \frac{1}{2}m\ell^2 + \frac{1}{2}\mathcal{J}\omega^2.$$

The conservation of $\hat{\mathcal{E}}$ up to the first collision is a simple energy estimate; we refer to [10] for such a result in a wider context. Since Theorem 1 establishes that, up to the first collision, the systems (1.1) and (1.18) are equivalent, in order to prove that $\mathcal{E}$ defined in (2.21) is conserved for solutions of (1.18), it is therefore sufficient to prove that, when $q\in\mathcal{Q}$, $\hat{\mathcal{E}}$ coincides with $\mathcal{E}$.

In order to prove this we use again the decomposition $u = u_1 + u_2$ (see (9.2)) so that

$$\begin{aligned} \hat{\mathcal{E}} &= \frac{1}{2}\int_{\mathcal{F}(q)} u_1^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathcal{F}(q)} u_2^2 \,\mathrm{d}x + \int_{\mathcal{F}(q)} u_1 \cdot u_2 \,\mathrm{d}x + \frac{1}{2}M_g p \cdot p, \\ &= \frac{1}{2}\int_{\mathcal{F}(q)} u_2^2 \,\mathrm{d}x + \int_{\mathcal{F}(q)} u_1 \cdot u_2 \,\mathrm{d}x + \frac{1}{2}M(q)p \cdot p, \end{aligned} \tag{9.61}$$

thanks to (9.26) and (1.9). Moreover we have that

$$\frac{1}{2}\int_{\mathcal{F}(q)} u_2^2 \,\mathrm{d}x = \frac{1}{2}\gamma^2 \int_{\mathcal{F}(q)} \nabla^\perp \psi \cdot \nabla^\perp \psi \,\mathrm{d}x = -\frac{1}{2}\gamma^2 C(q), \tag{9.62}$$

by an integration by parts, and

$$\int_{\mathcal{F}(q)} u_1 \cdot u_2 \,\mathrm{d}x = 0, \tag{9.63}$$

by another integration by parts. Gathering (9.61), (9.62) and (9.63) leads to $\hat{\mathcal{E}} = \mathcal{E}$, which concludes the proof of Proposition 1. $\square$

9.6.2. *Second proof: with the ODE formulation.* Let us give an alternative proof of Proposition 1 which only uses the ODE formulation (1.18).

We start with the observation that the energy $\mathcal{E}(q,p)$ as defined in (2.21) has for time derivative

$$\big(\mathcal{E}(q,p)\big)' = M(q)p' \cdot p + \frac{1}{2}(DM(q) \cdot p)p \cdot p - \frac{1}{2}\gamma^2 DC(q) \cdot p. \tag{9.64}$$

Now, thanks to (1.18) and (1.15c), we have

$$M(q)p' \cdot p = -\langle \Gamma(q), p, p \rangle \cdot p + F(q,p) \cdot p, \tag{9.65}$$

and

$$F(q,p) \cdot p = \gamma^2 E(q) \cdot p. \tag{9.66}$$

We introduce the matrix

$$S(q,p) := \Bigg(\sum_{1 \leqslant i \leqslant 3} \Gamma^k_{i,j}(q) p_i \Bigg)_{1 \leqslant k,j \leqslant 3}, \tag{9.67}$$

so that

$$\langle \Gamma(q), p, p \rangle = S(q,p)p. \tag{9.68}$$

Combining (9.64), (9.65), (9.66), (9.67) and (9.68) we obtain

$$\big(\mathcal{E}(q,p)\big)' = \gamma^2 \big(E(q) - \frac{1}{2}DC(q)\big) \cdot p + \big(\frac{1}{2}DM(q) \cdot p - S(q,p)\big)p \cdot p. \tag{9.69}$$

The first term of the right hand side vanishes thanks to (2.22).

The proof of Proposition 1 follows then from the following result.

Lemma 43. *For any* $(q,p) \in \mathcal{Q} \times \mathbb{R}^3$,

$$\frac{1}{2}DM(q) \cdot p - S(q,p) \text{ is skew-symmetric.} \tag{9.70}$$

$\square$

Proof of Lemma 43. We start with the observation that $DM(q) \cdot p$ is the 3×3 matrix containing the entries

$$\sum_{1 \leqslant k \leqslant 3} (M_a)^k_{i,j}(q)\, p_k, \text{ for } 1 \leqslant i,j \leqslant 3,$$

where $(M_a)^k_{i,j}(q)$ is defined in (1.11c).

On the other hand, the 3×3 matrix $S(q,p)$ contains the entries

$$\frac{1}{2} \sum_{1 \leqslant k \leqslant 3} \Big((M_a)^k_{i,j} + (M_a)^j_{i,k} - (M_a)^i_{k,j} \Big)(q)\, p_k,$$

for $1 \leqslant i,j \leqslant 3$. Therefore, the 3×3 matrix $DM(q) \cdot p - S(q,p)$ contains the entries

$$c_{ij}(q,p) = -\frac{1}{2} \sum_{1 \leqslant k \leqslant 3} \Big((M_a)^j_{i,k} - (M_a)^i_{k,j} \Big)(q)\, p_k,$$

for $1 \leqslant i,j \leqslant 3$. Using that the matrix $M(q)$ is symmetric, we get that $c_{ij}(q,p) = -c_{ji}(q,p)$ for $1 \leqslant i,j \leqslant 3$, which proves (b). $\square$

Appendix: Proof of Lemma 2

Let us first consider a smooth vector field $f := (f_1, f_2)$ on $\partial\mathcal{S}_0$ and observe that the complex integral $\int_{\partial\mathcal{S}_0} (f_1 - i f_2)\, \mathrm{d}z$ can be obtained by the circulation and flux of the vector field f thanks to the formula:

$$\int_{\partial\mathcal{S}_0} (f_1 - i f_2)\, \mathrm{d}z = \int_{\partial\mathcal{S}_0} \big(f \cdot \tau - i f \cdot n \big)\, ds. \tag{9.71}$$

In order to prove this, denote by $\gamma := (\gamma_1, \gamma_2)$ a parametrization of $\partial\mathcal{S}_0$ such that $\tau = (\gamma_1', \gamma_2')$, then $dz = (\gamma_1'(s) + i\gamma_2'(s))ds$ and $n = (-\gamma_2', \gamma_1')$. Hence (9.71) follows from

$$\int_{\partial\mathcal{S}_0} (f_1 - i f_2)\, \mathrm{d}z = \int_{\partial\mathcal{S}_0} (f_1\gamma_1' + f_2\gamma_2')\, ds - i \int_{\partial\mathcal{S}_0} (-f_1\gamma_2' + f_2\gamma_1')\, ds.$$

Now observe that $z(H_1 - iH_2) = f_1 - i f_2$ with $f_1 = x \cdot \nabla^\perp \psi^{-1}_{\partial\Omega}$ and $f_2 = x^\perp \cdot \nabla^\perp \psi^{-1}_{\partial\Omega}$ so that applying (9.71) we have that

$$\int_{\partial\mathcal{S}_0} z(H_1 - iH_2)\, \mathrm{d}z = \int_{\partial\mathcal{S}_0} g_{\partial\Omega}\, ds,$$

where, for $x \in \partial\mathcal{S}_0$,

$$g_{\partial\Omega}(x) := \begin{pmatrix} x \cdot \nabla^\perp \psi^{-1}_{\partial\Omega}(x) \\ x^\perp \cdot \nabla^\perp \psi^{-1}_{\partial\Omega}(x) \end{pmatrix} \cdot \tau - i \begin{pmatrix} x \cdot \nabla^\perp \psi^{-1}_{\partial\Omega}(x) \\ x^\perp \cdot \nabla^\perp \psi^{-1}_{\partial\Omega}(x) \end{pmatrix} \cdot n.$$

Moreover we have

$$\begin{aligned} g_{\partial\Omega} &= (x_1H_1 + x_2H_2)\tau_1 + (-x_2H_1 + x_1H_2)\tau_2 - i(x_1H_1 + x_2H_2)n_1 - i(-x_2H_1 + x_1H_2)n_2 \\ &= x_1(H_1\tau_1 + H_2\tau_2) + x_2(H_2\tau_1 - H_1\tau_2) - ix_1(H_1n_1 + H_2n_2) - ix_2(H_2n_1 - H_1n_2), \end{aligned}$$

and using that $(n_1, n_2) = (-\tau_2, \tau_1)$, we deduce that

$$g_{\partial\Omega} = z(\nabla^\perp \psi^{-1}_{\partial\Omega} \cdot \tau) - iz(\nabla^\perp \psi^{-1}_{\partial\Omega} \cdot n).$$

It is then sufficient to recall that $\nabla^\perp \psi^{-1}_{\partial\Omega} \cdot \tau = -\frac{\partial \psi^{-1}_{\partial\Omega}}{\partial n}$ and $\nabla^\perp \psi^{-1}_{\partial\Omega} \cdot n = 0$ to conclude that

$$\int_{\partial\mathcal{S}_0} z(H_1 - iH_2)\, \mathrm{d}z = -\int_{\partial\mathcal{S}_0} (x_1 + ix_2) \frac{\partial \psi^{-1}_{\partial\Omega}}{\partial n}\, \mathrm{d}s = \zeta_1 + i\zeta_2.$$

$\square$

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